Solutions, homework assignment 3

Section 3.3, 50: (a) The range of the $n \times n$ identity matrix I_n is all of $\mathbf{R}^n$: for any $\mathbf{y} \in \mathbf{R}^n$, if I let $\mathbf{x} = \mathbf{y}$, then $I_n\mathbf{x} = \mathbf{y}$.

The null space of I_n is $\{\mathbf{0}\}$: since $I_n\mathbf{x} = \mathbf{x}$ for any $\mathbf{x}$ in $\mathbf{R}^n$, the only solution to $I_n\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

(b) The $n \times n$ zero matrix O has the property that $O\mathbf{x} = \mathbf{0}$ for any vector $\mathbf{x}$ in $\mathbf{R}^n$. Therefore its range is $\{\mathbf{0}\}$, because $\mathbf{0}$ is the only thing which can appear on the right side of the equation $O\mathbf{x} = \mathbf{y}$, and its null space is all of $\mathbf{R}^n$.

(c) If A is an $n \times n$ nonsingular matrix, then its range is all of $\mathbf{R}^n$, and its null space is $\{\mathbf{0}\}$. I'll provide two explanations. First, since A is nonsingular, the only solution to the equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$ – that's the definition of nonsingular. This means that the only vector in the null space of A is the zero vector.

Also, Theorem 13 in Section 1.7 says that for any $n \times 1$ vector $\mathbf{b}$, there is a unique solution to $A\mathbf{x} = \mathbf{b}$. That is, for any vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ is consistent. Therefore every $n \times 1$ column vector $\mathbf{b}$ is in the range, so the range is all of $\mathbf{R}^n$.

The second explanation: I'll use the fact that since A is nonsingular, it is invertible. Suppose that $\mathbf{y}$ is in $\mathbf{R}^n$. Let $\mathbf{x} = A^{-1}\mathbf{y}$. Then

$$A\mathbf{x} = A(A^{-1}\mathbf{y}) = (AA^{-1})\mathbf{y} = I_n\mathbf{y} = \mathbf{y}.$$

Thus $\mathbf{y}$ is in the range of A. Also, if $\mathbf{x}$ is in the null space of A, then

$$\mathbf{x} = I_n\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x})$$

Since $\mathbf{x}$ is in the null space, $A\mathbf{x} = \mathbf{0}$, so

$$\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{x} = \mathbf{0}$: the only vector in the null space of A is the zero vector.

Section 3.3, 52: (a) Suppose that $\mathbf{x}$ is in $\mathcal{N}(B)$, the null space of B. This means that $B\mathbf{x} = \mathbf{0}$. I want to show that $(AB)\mathbf{x} = \mathbf{0}$. Well,

$$(AB)\mathbf{x} = A(B\mathbf{x}),$$

but $B\mathbf{x} = \mathbf{0}$. Therefore $(AB)\mathbf{x} = \mathbf{0}$, which means that $\mathbf{x}$ is in $\mathcal{N}(AB)$.

(b) Suppose that $\mathbf{y}$ is in $\mathscr{R}(AB)$, the range of AB. Then there is a vector $\mathbf{x}$ such that $(AB)\mathbf{x} = \mathbf{y}$. Let $\mathbf{z} = B\mathbf{x}$. Then $A\mathbf{z} = \mathbf{y}$, which means that $\mathbf{y}$ is in the range of A, as desired.