Name: Answers

**Instructions**: This is a closed book exam, no calculators allowed. You may use one two-sided sheet of handwritten notes. Please check your answers carefully; I will only award limited partial credit. If you need more room, use the backs of the pages, and indicate that you have done so. Beware the Ides of March.

1. (10 points) Solve the system of equations

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & & & = & 0 \\ x_1 & - & x_2 & + & 2x_3 & = & 0 \\ -2x_1 & + & 3x_2 & - & 3x_3 & = & 10 \end{array}$$

For full credit, use Gaussian elimination (= row reduction).

**Solution:** I'll row-reduce the corresponding augmented matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ -2 & 3 & -3 & 10 \end{bmatrix} \xrightarrow{\mathbf{R}_1 \leftrightarrow \mathbf{R}_2} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & 3 & -3 & 10 \end{bmatrix} \xrightarrow{\mathbf{R}_2 - 2\mathbf{R}_1, \mathbf{R}_3 + 2\mathbf{R}_1} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & 1 & 10 \end{bmatrix}$$

$$\xrightarrow{\mathbf{R}_1 + \mathbf{R}_2, \mathbf{R}_3 - \mathbf{R}_2} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 5 & 10 \end{bmatrix} \xrightarrow{\frac{1}{5}\mathbf{R}_3} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\mathbf{R}_1 + 2\mathbf{R}_3, \mathbf{R}_2 + 4\mathbf{R}_3} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now I can read off the answer: $x_1 = 4$, $x_2 = 8$, and $x_3 = 2$. (It is easy to now plug these into the original system and thus verify that it is a solution.)

2. (10 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ -2 & 1 & 1 \end{bmatrix}$.

(a) What are the eigenvalues for $A$? [Hint: one of them is $\lambda = 2$. If you have found a characteristic polynomial for which 2 is not a root, you've made a mistake.]

**Solution:** The characteristic polynomial is

$$\begin{aligned} \begin{vmatrix} 1-t & 1 & 1 \\ 0 & 3-t & 3 \\ -2 & 1 & 1-t \end{vmatrix} &= (1-t)\begin{vmatrix} 3-t & 3 \\ 1 & 1-t \end{vmatrix} + (-2)\begin{vmatrix} 1 & 1 \\ 3-t & 3 \end{vmatrix} \quad \text{expand along first column} \\ &= (1-t)((3-t)(1-t)-3) - 2(3-(3-t)) \\ &= (1-t)(t^2-4t) - 2t \\ &= -t^3 + 5t^2 - 6t \\ &= -t(t^2 - 5t + 6) \\ &= -t(t-2)(t-3). \end{aligned}$$

The eigenvalues are the roots of this, which are 0, 2, and 3.

(b) Let $\lambda$ be one of the eigenvalues (your choice), and find a basis for its eigenspace $E_\lambda$.

**Solution:** You only need to do one of these, but I'll do all three.

$\lambda = 0$: the eigenspace is the null space of $A - 0I = A$. That is, we solve the system $A\mathbf{x} = \mathbf{0}$, which can be done by row-reducing the corresponding augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ -2 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\mathbf{R}_3 + 2\mathbf{R}_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This translates into: $x_1 = 0$ and $x_2 + x_3 = 0$, so $x_2 = -x_3$. So one parameter, $x_3$, determines each such vector, and a basis is $\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T$.

$\lambda = 2$: the eigenspace is the null space of $A - 2I = A$. That is, we solve the system $(A - 2I)\mathbf{x} = \mathbf{0}$, which can be done by row-reducing the corresponding augmented matrix

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ -2 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This translates into $x_1 = -2x_3$ and $x_2 = -3x_3$, so one parameter, $x_3$, determines each such vector, and a basis is $\begin{bmatrix} -2 & -3 & 1 \end{bmatrix}^T$.

$\lambda = 3$: the eigenspace is the null space of $A - 3I = A$. That is, we solve the system $(A - 3I)\mathbf{x} = \mathbf{0}$, which can be done by row-reducing the corresponding augmented matrix

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ -2 & 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This translates into $x_1 = x_2/2$ and $x_3 = 0$, so one parameter, $x_2$, determines each such vector, and a basis is $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$.
Whichever of these you choose, it's not hard to check your answer: multiply $A$ by your eigenvector candidate, and see if you get $\lambda\mathbf{x}$.

3. (10 points) Suppose that you wanted to find the least-squares linear fit to the following data:

| $t$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| $y$ | 0 | 1 | 2 | 4 | 5 | 7 |

Tell me what computations you would carry out. DO NOT ACTUALLY DO THE COMPUTATIONS: just tell me the relevant matrices and describe the least-squares method for the given data.

**Solution:** Let $A$ be the matrix formed by putting 1s in the first column and the $t$ values in the second column, and let $\mathbf{y}$ be the vector of $y$ values:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 5 \\ 7 \end{bmatrix}.$$

The first thing to check is whether the system

$$A \begin{bmatrix} b \\ m \end{bmatrix} = \mathbf{y}$$

has a solution $\begin{bmatrix} b \\ m \end{bmatrix}$. If so, then all of the given points lie on the line $y = mt + b$. If not, then the system is inconsistent, and the method of least-squares is needed. The best least-squares linear fit is the solution to the system

$$A^T A \begin{bmatrix} b \\ m \end{bmatrix} = A^T \mathbf{y}.$$

(So: compute $A^T A$, compute $A^T\mathbf{y}$, and solve this system of equations by using row-reduction or any other method that you like.)

4. (15 points) Suppose that $A$ is an $n \times n$ matrix.

(a) Suppose that $\lambda$ is a scalar and $\mathbf{x}$ is a vector, and that $A\mathbf{x} = \lambda\mathbf{x}$. Explain why $A^2\mathbf{x} = \lambda^2\mathbf{x}$.

**Solution:** Compute $A^2\mathbf{x}$:

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}.$$

(b) For this part, suppose that $n = 2$, and suppose that $A\begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}-6\\-12\end{bmatrix}$. What is $A^2\begin{bmatrix}1\\2\end{bmatrix}$?

**Solution:** If $A\begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}-6\\-12\end{bmatrix}$, then $\begin{bmatrix}3\\6\end{bmatrix}$ is an eigenvector with eigenvalue $-2$. Therefore so is any scalar multiple of it: $\begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector with eigenvalue $-2$, so

$$A\begin{bmatrix}1\\2\end{bmatrix} = -2\begin{bmatrix}1\\2\end{bmatrix}.$$

Therefore by part (a),

$$A^2\begin{bmatrix}1\\2\end{bmatrix} = (-2)^2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4\\8\end{bmatrix}.$$

(c) Suppose that $A$ is an $n \times n$ matrix such that $A^2$ is the identity matrix: $A^2 = I_n$. What are the possible eigenvalues for $A$? Explain your answer.

**Solution:** Suppose that $\lambda$ is an eigenvalue for $A$. Then there is a vector $\mathbf{x}$ so that $A\mathbf{x} = \lambda\mathbf{x}$. Now compute $A^2\mathbf{x}$: by part (a), this equals $\lambda^2\mathbf{x}$. On the other hand, since $A^2 = I_n$, this equals $\mathbf{x}$. That is, $\lambda^2\mathbf{x} = \mathbf{x}$. Thus (since $\mathbf{x}$ must be nonzero), $\lambda^2 = 1$, so $\lambda = \pm 1$.

By the way, note that $A$ need not equal the identity matrix. For example, in the $2 \times 2$ case, $A$ could be the matrix $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$, or $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

5. (10 points) Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. Assume that $b \neq 0$. Find the eigenvalues of $A$, and for each eigenvalue, find the corresponding eigenvectors.

**Solution:** The characteristic polynomial of $A$ is

$$\det \begin{bmatrix} a-t & b \\ b & a-t \end{bmatrix} = (a-t)^2 - b^2 = t^2 - 2at + (a^2 - b^2).$$

This is a quadratic equation, and its roots are given by the quadratic formula:

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2 - b^2)}}{2} = \frac{2a \pm \sqrt{4b^2}}{2} = a \pm b.$$

So the eigenvalues are $a+b$ and $a-b$. Now find the eigenvectors:

For $\lambda = a+b$, the relevant equation is

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (a+b)x \\ (a+b)y \end{bmatrix}.$$

The matrix multiplication on the left side turns this into

$$\begin{bmatrix} ax+by \\ bx+ay \end{bmatrix} = \begin{bmatrix} (a+b)x \\ (a+b)y \end{bmatrix}.$$

The first coordinate says $ax+by = ax+bx$. Cancel the $ax$ summand to get $by = bx$. Divide both sides by $b$ (this is okay since we're told that $b \neq 0$): $x = y$. The second coordinate gives the same thing. Thus the eigenvectors are all of the form $\begin{bmatrix} x \\ x \end{bmatrix}$. One particular eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = a-b$, the relevant equation is

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (a-b)x \\ (a-b)y \end{bmatrix}.$$

The matrix multiplication on the left side turns this into

$$\begin{bmatrix} ax+by \\ bx+ay \end{bmatrix} = \begin{bmatrix} (a-b)x \\ (a-b)y \end{bmatrix}.$$

The first coordinate says $ax+by = ax-bx$. Cancel the $ax$ summand to get $by = -bx$. Divide both sides by $b$ (this is okay since we're told that $b \neq 0$): $y = -x$. The second coordinate gives the same thing. Thus the eigenvectors are all of the form $\begin{bmatrix} x \\ -x \end{bmatrix}$. One particular eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

6. (15 points) Let $A = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}$.

   (a) Find a basis for the null space of $A$.

**Solution:** Row reduce the corresponding augmented matrix:

$$\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 0 \\ 2 & 4 & 6 & 8 & 10 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 0 \\ 0 & -2 & -4 & -6 & -8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \end{bmatrix}.$$

This translates into

$$x_1 - x_3 - 2x_4 - 3x_5 = 0,$$
$$x_2 + 2x_3 + 3x_4 + 4x_5 = 0,$$

or

$$x_1 = x_3 + 2x_4 + 3x_5,$$
$$x_2 = -2x_3 - 3x_4 - 4x_5.$$

Therefore we need the three parameters $x_3$, $x_4$, and $x_5$, and the vectors in the null space look like

$$\begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus a basis is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   (b) Compute the rank of $A$ and the nullity of $A$.

**Solution:** Since there are three vectors in a basis for the null space, the nullity of $A$ is 3. Since the rank and nullity sum to 5 (the number of columns in $A$, then rank of $A$ is 2.

7. (10 points) Suppose that $T : \mathbf{R}^n \to \mathbf{R}^k$ is a linear transformation, and suppose that the nullity of $T$ is 0. Prove that $T$ has the following property:

   For any vectors $\mathbf{w}$ and $\mathbf{x}$ in $\mathbf{R}^n$, if $T(\mathbf{w}) = T(\mathbf{x})$, then $\mathbf{w} = \mathbf{x}$.

   (That is, show that $T$ is *one-to-one*, or *injective*.)

**Solution:** Suppose that $\mathbf{w}$ and $\mathbf{x}$ are vectors for which $T(\mathbf{w}) = T(\mathbf{x})$. Then

$$T(\mathbf{w}) - T(\mathbf{x}) = \mathbf{0},$$

which means that

$$T(\mathbf{w} - \mathbf{x}) = \mathbf{0}.$$

Thus $\mathbf{w} - \mathbf{x}$ is in the null space of $T$. Since the nullity of $T$ is zero, the zero vector is the only vector in the null space of $T$. Therefore $\mathbf{w} - \mathbf{x} = \mathbf{0}$, which means that $\mathbf{w} = \mathbf{x}$.