Math 125 B and C **Midterm 2** **Palmieri, Autumn 2006**

Your Name

Your Signature

Student ID #

TA's Name and quiz section (circle):

Cady		Cruz		Jacobs	
BA	CB	BB	BC	CA	CC

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8\frac{1}{2}$" $\times$ 11" sheet of handwritten notes (one side).
- Graphing calculators are not allowed.
- Give your answers in exact form, not decimals, except where indicated.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct. You may use any of the 20 integrals from the table on p. 506 of the text without deriving them. Show your work in evaluating any other integrals, even if they are on your note sheet.
- **Check your work carefully.** We will award only limited partial credit.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 5 pages, plus this cover sheet. Make sure that your exam is complete.

Question	Points	Score
1	16	
2	8	
3	8	
4	8	
5	10	
Total	50	

1. (a) (8 points) Compute $\displaystyle\int \frac{\sin(3t)\cos(3t)}{\cos^2(3t) - 3\cos(3t) + 2}\,dt$.

Solution: (Similar to problem 1 on the week 7 worksheet.)

First, make a substitution: let $u = \cos(3t)$, so $du = -3\sin(3t)\,dt$, so $\sin(3t)\,dt = -\frac{1}{3}\,du$. Then the integral becomes

$$-\frac{1}{3}\int \frac{u}{u^2 - 3u + 2}\,du = -\frac{1}{3}\int \frac{u}{(u-1)(u-2)}\,du$$

Now use partial fractions:

$$\begin{aligned}
&= -\frac{1}{3}\int \left(\frac{A}{u-1} + \frac{B}{u-2}\right) du \\
&= -\frac{1}{3}\int \left(\frac{-1}{u-1} + \frac{2}{u-2}\right) du \\
&= -\frac{1}{3}\left(-\ln|u-1| + 2\ln|u-2|\right) + C \\
&= \boxed{\frac{1}{3}\ln|\cos(3t) - 1| - \frac{2}{3}\ln|\cos(3t) - 2| + C}.
\end{aligned}$$

The absolute value signs are important here, because for example, $\cos(3t) - 2$ is always negative, so $\ln(\cos(3t) - 2)$ is not defined.

(b) (8 points) Compute the average value of $\cos^3(x)$ on the interval $[0, \pi/2]$.

Solution: (Basic trig integral + average value.)

The average value is given by the trig integral

$$A = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos^3(x)\,dx$$

Since cos appears to an odd power, let $u = \sin(x)$, so $du = \cos x$. When $x = 0$, $u = \sin(0) = 0$, and when $x = \pi/2$, $u = \sin(\pi/2) = 1$.

$$\begin{aligned}
&= \frac{2}{\pi}\int_0^1 (1 - u^2)\,du = \frac{2}{\pi}\left[u - \frac{1}{3}u^3\right]_0^1 \\
&= \frac{2}{\pi}\,\frac{2}{3} = \boxed{\frac{4}{3\pi}}.
\end{aligned}$$

2. (8 points) Compute $\displaystyle\int_{-2}^{2} \frac{1}{\sqrt{x^2+4x+8}}\,dx.$

Solution: First complete the square, then do a substitution, then a trig substitution:

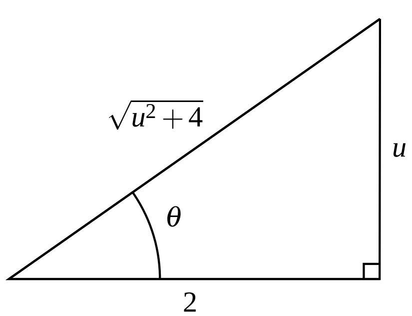
$$\int_{-2}^{2} \frac{1}{\sqrt{x^2+4x+8}}\,dx = \int_{-2}^{2} \frac{1}{\sqrt{(x+2)^2+4}}\,dx$$

Let $u = x+2$. Change the limits of integration accordingly.

$$= \int_{0}^{4} \frac{1}{\sqrt{u^2+4}}\,du$$

Let $u = 2\tan\theta$, so $du = 2\sec^2\theta\,d\theta$. See picture of triangle below.

$$\begin{aligned}
&= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta + \tan\theta| \\
&= \ln\left|\frac{\sqrt{u^2+4}}{2} + \frac{u}{2}\right|_0^4 \\
&= \ln\left|\frac{\sqrt{20}}{2} + \frac{4}{2}\right| - \ln\left|\frac{\sqrt{4}}{2}\right| \\
&= \boxed{\ln(\sqrt{5}+2)}
\end{aligned}$$



3. (8 points) Determine if the improper integral $\int_1^\infty xe^{-3x}\,dx$ converges or diverges. If it converges, evaluate it.

Solution: (Similar to problem 19 in Section 7.4.)

$$\int_1^\infty xe^{-3x}\,dx = \lim_{b\to\infty}\left(\int_1^b xe^{-3x}\,dx\right)$$

Integration by parts: $u = x$, $dv = e^{-3x}\,dx$, so $du = dx$ and $v = -\frac{1}{3}e^{-3x}$:

$$\begin{aligned}
&= \lim_{b\to\infty}\left(-\frac{1}{3}xe^{-3x}\Big|_1^b + \frac{1}{3}\int_1^b e^{-3x}\,dx\right)\\
&= \lim_{b\to\infty}\left(-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right)_1^b\\
&= \lim_{b\to\infty}\left[\left(-\frac{1}{3}be^{-3b} - \frac{1}{9}e^{-3b}\right) - \left(-\frac{1}{3}e^{-3} - \frac{1}{9}e^{-3}\right)\right]
\end{aligned}$$

As $b \to \infty$, $e^{-3b} \to 0$ (basic limit) and $be^{-3b} \to 0$ (by L'Hôpital's rule). Thus the integral converges, and equals

$$= \frac{1}{3}e^{-3} + \frac{1}{9}e^{-3} = \boxed{\frac{4}{9}e^{-3}}.$$

A slight variant on evaluating the integral: first make the substitution $w = -3x$, $x = -\frac{1}{3}w$, $dw = -3\,dx$, and $dx = -\frac{1}{3}dw$. Then the substitution has this effect:

$$\lim_{b\to\infty}\left(\int_1^b xe^{-3x}\,dx\right) = \lim_{b\to\infty}\left(\int (-\frac{1}{3})we^w(-\frac{1}{3})dw\right) = \lim_{b\to\infty}\frac{1}{9}\left(\int we^w dw\right).$$

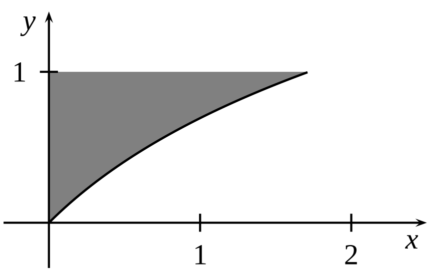
Now do integration by parts as above, then change back to x at the end, before plugging in the endpoints. Alternatively, change the endpoints when you do the substitution: when $x = 1$, then $w = -3$, and when $x = b$, $w = -3b$, so you would work with

$$= \lim_{b\to\infty}\frac{1}{9}\left(\int_{-3}^{-3b} we^w dw\right).$$

4. (8 points) A spring has natural length of 30 cm (= 0.3 meters). It requires 2 J of work to stretch it from 40 cm to 45 cm. How far beyond its natural length will a force of 64 N keep the spring stretched?

> **Solution:** (Like other spring problems in Section 6.4.)
>
> Since the force required to hold a spring at distance x from its natural length is $f(x) = kx$ (that's *Hooke's law*), the work required to stretch a spring from length a to length b is $W = \int_a^b kx\,dx = \frac{1}{2}kx^2\Big|_a^b = \frac{1}{2}kb^2 - \frac{1}{2}ka^2$. In the second sentence of the problem, $a = 0.4 - 0.3 = 0.1$ and $b = 0.45 - 0.3 = 0.15$ (because distances are measured in meters from the natural length of the spring) while $W = 2$, so we get the equation
>
> $$2 = \frac{1}{2}k(0.15^2 - 0.1^2) = \frac{1}{2}k(0.0125).$$
>
> Thus $k = 4/0.0125 = 320$.
>
> Now apply a force of 64 N: the force formula is $f(x) = kx$, which becomes $64 = 320x$. Thus $x = 64/320 = 1/5 = \boxed{0.2\,\mathrm{m}} = \boxed{20\,\mathrm{cm}}$.

5. A portion of the graph of $y = \ln(x+1)$ between $x = 0$ and $x = e - 1$ is rotated around the y-axis to form a container. The container is filled with water. Distance is measured in meters and the density of water is 1000 kg/m^3.



(a) (6 points) Set up, but DO NOT EVALUATE, an integral to compute the work required to pump the water out over the side (which is at height $y = 1$).

> **Solution:** (This is very similar to problem 3 on the practice problems for the week of the midterm.)
>
> Since this is a pumping problem, we should use *horizontal* slices, and so we will get a dy integral. So I want to solve for x in terms of y, to find the radius of the container. Since $y = \ln(x+1)$, exponentiate both sides to get
>
> $$e^y = x + 1 \qquad \text{so} \qquad x = e^y - 1.$$
>
> When $x = 0$, $y = 0$, and when $x = e - 1$, $y = 1$. So y ranges from 0 to 1 in this problem.
>
> Given a "slice" of water at height y with thickness dy, its weight is
>
> $$\begin{aligned} g \times \text{mass} &= g \times \text{density} \times \text{volume} \\ &= 1000g \times \text{volume}. \end{aligned}$$

Now, the volume is $\pi(\text{radius})^2 dy = \pi(e^y - 1)^2\,dy$, so the weight is

$$1000g\pi(e^y-1)^2\,dy.$$

This must be lifted a distance of $1-y$, to get from height y to height 1 at the top of the container, so the work required to lift this slice is

$$\text{weight} \times \text{distance} = 1000g\pi(e^y-1)^2(1-y)\,dy.$$

Now integrate as y goes from 0 to 1:

$$\boxed{W = \int_0^1 1000g\pi(e^y-1)^2(1-y)\,dy = 1000\pi g\int_0^1 (e^y-1)^2(1-y)\,dy}.$$

(b) (4 points) Use $n=4$ subdivisions and the midpoint rule to approximate the value of the integral in part (a). Give an answer correct to at least two significant digits.

Solution: Divide the interval $[0,1]$ into four equal pieces; thus $\Delta y = 0.25$. The midpoints of the pieces are 0.125, 0.375, 0.625, and 0.875, so plug these into the function being integrated, and add up the results:

$$\begin{aligned} W &\approx 0.25 \times 1000\pi g\Big((e^{0.125}-1)^2(1-0.125)+(e^{0.375}-1)^2(1-0.375) \\ &\qquad + (e^{0.625}-1)^2(1-0.625)+(e^{0.875}-1)^2(1-0.875)\Big) \\ &\approx 2450\pi(0.672) \approx 5174 \approx \boxed{5,200\,\mathrm{J}}, \end{aligned}$$

to two significant digits. (This is using $g = 9.8\ \mathrm{m/sec}^2$. If we want better accuracy, we need to know g more accurately.)