Math 125 B and C **Midterm 1** **Palmieri, Autumn 2006**

Your Name

Your Signature

Student ID #

TA's Name and quiz section (circle):

Cady		Cruz		Jacobs	
BA	CB	BB	BC	CA	CC

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one $8\frac{1}{2}$" $\times$ 11" sheet of handwritten notes (one side).
- Graphing calculators are not allowed.
- Give your answers in exact form, not decimals.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- **Check your work carefully.** We will award only limited partial credit.
- Place a box around your answer to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 5 pages, plus this cover sheet. Make sure that your exam is complete.

Question	Points	Score
1	14	
2	11	
3	7	
4	10	
5	8	
Total	50	

1. (a) (7 points) Compute $\int \left(3x^4 - \frac{1}{x} + 5\cos(x)\right) dx$.

Solution: No substitutions required: these are all easy antiderivatives. The answer is

$$\boxed{\frac{3}{5}x^5 - \ln|x| + 5\sin x + C}.$$

Check: the derivative of this is

$$\frac{3}{5}5x^4 - \frac{1}{x} + 5\cos x,$$

which is what it's supposed to be.

(b) (7 points) Compute $\int \sec^2(2x)\tan^5(2x)\,dx$.

Solution: Make the substitution $u = \tan(2x)$. Then $du = 2\sec^2(2x)dx$, and the integral becomes

$$\int u^5 \frac{1}{2}\,du = \frac{1}{2}\frac{1}{6}u^6 + C.$$

Now substitute back in for u: the answer is

$$\boxed{\frac{1}{12}\tan^6(2x) + C}.$$

Check: the derivative of this is (by the chain rule)

$$\frac{1}{12}6 \times 2\tan^5(2x)\sec^2(2x),$$

which is the original function.

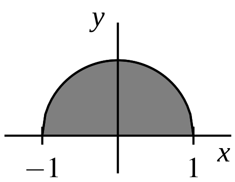
Alternatively, make the substitution $u = \sec(2x)$. Then $du = 2\sec(2x)\tan(2x)$, so $\sec^2(2x)\tan(2x)dx = \frac{1}{2}u\,du$. This leaves a factor of $\tan^4(2x)$ to deal with, at which point I can appeal to the trig identity $\sec^2(\theta) = 1 + \tan^2(\theta)$, or $\tan^2(\theta) = \sec^2(\theta) - 1$, so $\tan^4(2x) = (\tan^2(2x))^2 = (\sec^2(2x) - 1)^2$. So the integral is

$$\begin{aligned}\int \sec(2x)(\sec^2(2x) - 1)^2 \sec(2x)\tan(2x)\,dx &= \frac{1}{2}\int u(u^2-1)^2\,du = \frac{1}{2}\int u(u^4 - 2u^2 + 1)\,du \\ &= \int \frac{1}{2}\int (u^5 - 2u^3 + u)\,du = \frac{1}{2}\left(\frac{1}{6}u^6 - \frac{2}{4}u^4 + \frac{1}{2}u^2\right) + C \\ &= \frac{1}{12}\sec^6(2x) - \frac{1}{4}\sec^4(2x) + \frac{1}{4}\sec^2(2x) + C.\end{aligned}$$

This is actually equal to $\frac{1}{12}\tan^6(2x) + C'$ by the same trig identity (note, thought, that the constants are not the same).

2. (a) (4 points) Compute $\int_{-1}^{1} \sqrt{1-x^2}\,dx$. [Hint: interpret the integral as an area.]

Solution: This definite integral is the area under the curve $y = \sqrt{1-x^2}$, which is the top half of the circle $x^2+y^2=1$. Thus the area is half the area of this circle: the answer is $\boxed{\pi/2}$.



(b) (7 points) Compute $\int_{1}^{2} x(2-x)^7\,dx$.

Solution: Make the substitution $u = 2-x$. Then $du = -x\,dx$, and also $x = 2-u$, and for the endpoints, when $x=1$, $u=1$, and when $x=2$, $u=0$. So the integral becomes

$$-\int_{1}^{0}(2-u)u^7\,du = \int_0^1 (2u^7-u^8)\,du = \left[\frac{2}{8}u^8 - \frac{1}{9}u^9\right]_0^1 = \frac{1}{4}-\frac{1}{9} = \boxed{\frac{5}{36}}.$$

Alternatively, make the same substitution, but don't change the endpoints. In fact, don't worry about the endpoints until the very end:

$$\int x(2-x)^7\,dx = -\int(2-u)u^7\,du = \int(u^8-2u^7)\,du = \frac{1}{9}u^9 - \frac{2}{8}u^8 = \frac{1}{9}(2-x)^9 - \frac{1}{4}(2-x)^8.$$

Now plug in the original endpoints:

$$\left[\frac{1}{9}(2-x)^9 - \frac{1}{4}(2-x)^8\right]_1^2 = -\left(\frac{1}{9}-\frac{1}{4}\right) = \boxed{\frac{5}{36}}.$$

Alternatively, you could multiply out $x(2-x)^7$ and integrate it. This is unpleasant and easy to screw up, so it's not a very good method.

3. (7 points) Find the interval (or intervals) on which the curve

$$y = \int_2^{x^2-x} (1+\sin^2(t))\,dt$$

is increasing.

Solution: The curve is increasing where its derivative is positive, so I'll compute its derivative: by the first part of the Fundamental Theorem of Calculus,

$$y' = (2x-1)(1+\sin^2(x^2-x)).$$

In more detail: let $g(u) = \int_2^u (1+\sin^2(t))\,dt$. Then $g'(u) = 1+\sin^2(u)$, by the FTC. y is equal to $g(x^2-x)$, and by the chain rule,

$$y' = (x^2-x)'g'(x^2-x) = (2x-1)(1+\sin^2(x^2-x)).$$

Because it's squared, the term $\sin^2(x^2-x)$ is always greater than or equal to zero, so $1+\sin^2(x^2-x)$ is always positive. Thus y' is positive whenever $2x-1$ is, which is when $x > 1/2$. So

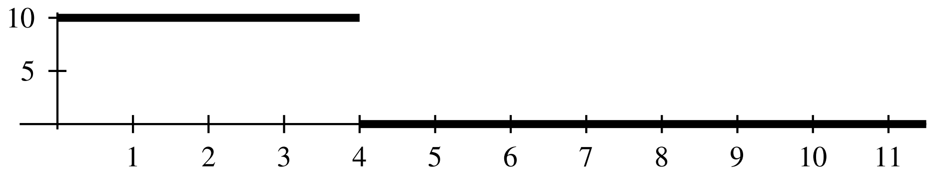
$\boxed{y \text{ is increasing when } x > 1/2}$.

4. A spaceship is at rest in space. At time $t = 0$, the pilot turns the engine on, and then turns it off when $t = 4$. As a result, the spaceship's acceleration is given by

$$a(t) = \begin{cases} 10, & \text{if } 0 \le t \le 4, \\ 0, & \text{if } t > 4. \end{cases}$$

(a) (5 points) What is the spaceship's velocity when $t = 2$? When $t = 4$? When $t = 10$?

Solution: Draw a graph of acceleration as a function of t; then the velocity at time t is the area under the acceleration curve between 0 and t. In this case, the acceleration curve looks like this:



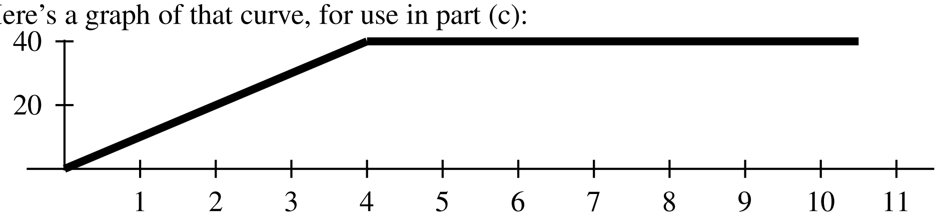
It's easy to compute areas under this curve: $\boxed{v(2) = 20}$, $\boxed{v(4) = 40}$, and $\boxed{v(10) = 40}$.

(b) (3 points) Find a formula for $v(t)$, the velocity of the spaceship, valid for all $t \ge 0$.

Solution: We just need a formula for the area under the acceleration curve between 0 and t. This formula is

$$\boxed{v(t) = \begin{cases} 10t & \text{if } 0 \le t \le 4, \\ 40 & \text{if } t > 4. \end{cases}}$$

Here's a graph of that curve, for use in part (c):

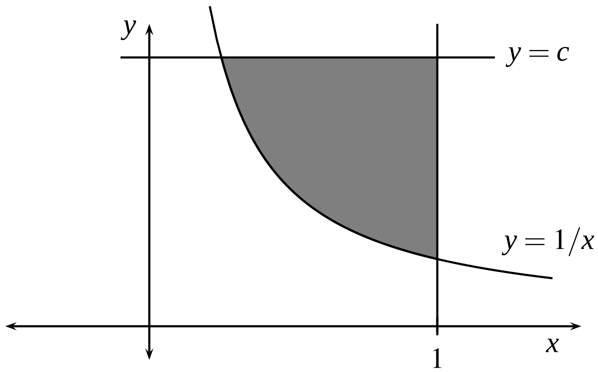


(c) (2 points) How far has the spaceship traveled after 10 seconds?

Solution: The distance traveled is the area under the velocity curve between $t = 0$ and $t = 10$. If you draw the velocity curve (as I've done in part (b)), you can easily compute the area: it's $\boxed{320}$.

5. (8 points) Consider the region bounded by the curve $y = 1/x$, the line $x = 1$, and the line $y = c$ for some constant $c > 1$. Rotate this region about the y-axis. For what value of c is the volume of the resulting solid equal to 2π?

Solution: Here's a picture of the region:



Now compute the volume of the resulting solid in terms of c, set it equal to 2π, and solve for c. You can use either washers or cylindrical shells.

Using washers: use horizontal slices and integrate with respect to y. y ranges from 1 to c, the outer radius of the washer is 1, and the inner radius is $x = 1/y$. So the volume is

$$V = \int_1^c \pi \left(1^2 - \left(\frac{1}{y}\right)^2\right) dy = \pi \left(y + \frac{1}{y}\right)_1^c$$
$$= \pi \left(c + \frac{1}{c} - 2\right).$$

Using cylindrical shells: use vertical slices and integrate with respect to x. x ranges from $1/c$ to 1, the radius of each shell is x, and the height of each cylindrical shell is $c - 1/x$. So the volume is

$$V = \int_{1/c}^1 2\pi x(c - 1/x)\, dx = 2\pi \int_{1/c}^1 (cx - 1)\, dx$$
$$= 2\pi \left(\frac{1}{2}cx^2 - x\right)_{1/c}^1 = 2\pi \left(\frac{c}{2} - 1 - \frac{1}{2c} + \frac{1}{c}\right)$$
$$= 2\pi \left(\frac{c}{2} - 1 + \frac{1}{2c}\right) = \pi \left(c - 2 + \frac{1}{c}\right).$$

Either way, the volume is $\pi\left(c - 2 + \frac{1}{c}\right)$. This is supposed to equal 2π:

$$2\pi = \pi \left(c - 2 + \frac{1}{c}\right).$$

Thus $2 = c - 2 + 1/c$. Multiply through by c to get a quadratic equation: $c^2 - 4c + 1 = 0$. By the quadratic formula, the roots of this are $c = \dfrac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$. The root $2 - \sqrt{3}$ is less than 1, and since c is supposed to be bigger than 1, the answer is $\boxed{c = 2 + \sqrt{3}}$.