Mathematics 403A Winter 2005

Theorem 1. *e is transcendental.*

I will need the following formulation of the Mean Value Theorem.

Theorem 2 (Mean Value Theorem). *If $g(x)$ is a continuously differential function on the interval $[x_1, x_2]$, then for some $\theta \in (0,1)$,*

$$\frac{g(x_1) - g(x_2)}{x_1 - x_2} = g'(x_1 + \theta(x_2 - x_1)).$$

Proof that e is transcendental. [This is from *Topics in Algebra* by Herstein.]

Given a polynomial $f(x) \in \mathbf{R}[x]$ of degree r, let

$$F(x) = f(x) + f'(x) + f''(x) + \cdots + f^{(r)}(x).$$

Then one can check that $\frac{d}{dx}(e^{-x}F(x)) = -e^{-x}f(x)$.

Apply the Mean Value Theorem to $e^{-x}F(x)$, on the interval $[0,k]$, for each positive integer k: there are numbers $\theta_k \in (0,1)$ such that

$$e^{-k}F(k) - F(0) = -ke^{(1-\theta_k)k} f(\theta_k k).$$

Also, define ε_k to be $\varepsilon_k = F(k) - e^k F(0)$. That is,

$$\begin{aligned}
\varepsilon_1 &= F(1) - eF(0) = -e^{(1-\theta_1)} f(\theta_1), \\
\varepsilon_2 &= F(2) - e^2F(0) = -2e^{2(1-\theta_2)} f(2\theta_2), \\
&\vdots \\
\varepsilon_n &= F(n) - e^nF(0) = -ne^{n(1-\theta_n)} f(n\theta_n).
\end{aligned}$$

Now suppose that e is algebraic: suppose there are integers c_i such that

$$c_n e^n + c_{n-1}e^{n-1} + \cdots + c_1 e + c_0 = 0.$$

We will (eventually) derive a contradiction.

Multiply the equation defining ε_i by c_i and add up the resulting equations:

$$c_1F(1) + c_2F(2) + \cdots + c_nF(n) - F(0)(c_1e + c_2e^2 + \cdots + c_ne^n) = c_1\varepsilon_1 + c_2\varepsilon_2 + \cdots + c_n\varepsilon_n.$$

The term in parentheses equals $-c_0$, so I can rewrite this as

$$c_0F(0) + c_1F(1) + c_2F(2) + \cdots + c_nF(n) = c_1\varepsilon_1 + c_2\varepsilon_2 + \cdots + c_n\varepsilon_n.$$

This holds for any $f(x)$; now we pick a particular one. Let p be a prime number such that $p > n$ and $p > c_0$. (Later on I'll assume that p is even larger, but this is good enough for now.) Let

$$f(x) = \frac{1}{(p-1)!} x^{p-1}(1-x)^p(2-x)^p \dots (n-x)^p.$$

Then

$$f(x) = \frac{(n!)^p}{(p-1)!} x^{p-1} + \frac{a_0}{(p-1)!} x^p + \frac{a_1}{(p-1)!} x^{p+1} + \dots$$

for some integers a_0, a_1, ...

Claim. For each $i \geq p$, $f^{(i)}(x)$ is a polynomial with integer coefficients, each of which is divisible by p.

Proof of claim. Exercise. □

For $j = 1, 2, \dots, n$, $f(j) = 0$ with multiplicity p, so $f^{(i)}(j) = 0$ when $i \leq p-1$. Thus

$$F(j) = \underbrace{f(j) + f'(j) + \dots + f^{(p-1)}(j)}_{} + \underbrace{f^{(p)}(j) + \dots + f^{(r)}(j)}_{}$$

is an integer which is divisible by p: the first bunch of terms are zero, and the second bunch are all divisible by p, by the claim.

On the other hand, consider

$$F(0) = \underbrace{f(0) + f'(0) + \dots + f^{(p-2)}(0)}_{} + f^{(p-1)}(0) + \underbrace{f^{(p)}(0) + \dots}_{}.$$

The first bunch of terms are all zero, and the last bunch are all divisible by p, but $f^{(p-1)}(0) = (n!)^p$. Since p was chosen to be larger than n, this is not divisible by p.

Also, p was chosen to be larger than c_0, and so doesn't divide c_0. As a result, p doesn't divide the integer $c_0F(0) + c_1F(1) + \dots + c_nF(n)$.

Recall from before that $c_0F(0) + c_1F(1) + c_2F(2) + \dots + c_nF(n) = c_1\varepsilon_1 + c_2\varepsilon_2 + \dots + c_n\varepsilon_n$. The left side of this equation is a nonnegative integer (since it is not divisible by p). I will show that the right side has absolute value less than 1, and this will be a contradiction.

Fix i with $1 \leq i \leq n$. From the definition of ε_i and the formula for $f(x)$, we see that

$$\varepsilon_i = -ie^{i(1-\theta_i)} f(i\theta_i) = \frac{-ie^{i(1-\theta_i)}(i\theta_i)^{p-1}(1-i\theta_1)^p \dots (n-i\theta_i)^p}{(p-1)!}.$$

Since $0 < \theta_i < 1$, we have

$$|\varepsilon_i| \leq \frac{e^n n^p (n!)^p}{(p-1)!}.$$

As p goes to ∞, the right hand side goes to zero, so for p a sufficiently large prime,

$$|c_1\varepsilon_1 + \dots + c_n\varepsilon_n| < 1.$$

But $c_1\varepsilon_1 + \dots + c_n\varepsilon_n$ must be a nonzero integer. This is the contradiction we were looking for. □