Some solutions for practice problems,  
Sections 11.2 and 11.3

11.2.1 (a) \( R[x,y] \) is not a Euclidean domain; in fact, it is not even a principal ideal domain. We do have a notion of degree, and we know that if \( p \) and \( q \) are polynomials and \( p \equiv q \), then the degree of \( p \) is at most the degree of \( q \). Given this, it’s easy to show that \( R[x,y] \) is not a principal ideal domain: the ideal \( \langle x, y \rangle \) generated by \( x \) and \( y \) is not principal. If it were, say with \( I = \langle p(x,y) \rangle \), then we would have \( p|x \) and \( p|y \). The only possibility is for \( p \) to equal 0, in which case \( (p) = (0) \), or for \( p \) to be of degree 0, i.e., a constant, in which case \( (p) = (1) \).

(b) The ideal \( \langle 2,x \rangle \) is not principal, so \( \mathbb{Z}[x] \) is not a principal ideal domain. Again, if \( (2,x) = (p) \), then we would have \( p|2 \), so \( p = \pm 1 \) or \( p = \pm 2 \). If \( p \neq \pm 1 \), then \( p = \pm 2 \), so \( p \) does not divide \( x \).

11.2.11 (a) Assume that \( a \) and \( b \) are associates, so that \( a = bc \) and \( b = ad \) for some \( c,d \). Then we have \( a = bc = ade \); since we are working in an integral domain, we can cancel. So we get \( 1 = dc \); hence \( c \) and \( d \) are both units.

The converse is quick: if \( a = bu \) where \( u \) is a unit, then clearly \( b|a \); on the other hand, we have \( au^{-1} = b \), so \( a|b \).

(b) This problem is stated a bit ambiguously. The easier way to interpret it is this: find a ring \( R \) and elements \( a \) and \( b \) of \( R \) so that \( a \) and \( b \) are associates, and there is a nonunit \( r \) so that \( a = br \). Here’s one example: work in \( \mathbb{Z}/10\mathbb{Z} \). The units here are \( 1,3,7,9 \). Then \( 4 \) and \( 8 \) are associates, since \( 4 \cdot 2 = 8 \) and \( 8 \cdot 8 = 4 \), but I’ve just expressed these elements as being non-unit multiples of each other. They are also unit multiples of each other: \( 4 \cdot 7 = 8 \) and \( 8 \cdot 3 = 4 \).

This leads to the second interpretation: find a ring \( R \) and elements \( a \) and \( b \) so that \( a \) and \( b \) are associates, but neither one is a unit multiple of the other. Here’s an example of this: let \( R = \mathbb{Z}[x]/(5x) \). Then the units are \( 1 \) and \( -1 \). The elements \( x \) and \( 2x \) are associates: certainly \( 2x \) is a multiple of \( x \); since \( 5x = 0 \), then \( x = 6x = 3 \cdot 2x \), so \( x \) is also a multiple of \( 2x \). The unit multiples of \( x \) are \( x \) and \( -x = 4x \); the unit multiples of \( 2x \) are \( 2x \) and \( -2x = 3x \). In particular, neither one is a unit multiple of the other.

11.3.1 Assume \( f(x) \) is irreducible; assume we have \( f(ax+b) = p(x)q(x) \) for some \( p,q \in F[x] \). Make the substitution \( y = ax + b \), i.e., \( x = a^{-1}(y - b) \). Then we have

\[
  f(y) = p(x)q(x) = p(a^{-1}(y - b))q(a^{-1}(y - b)).
\]

In other words, \( f(y) \) factors. So \( p(a^{-1}(y - b)) \) must be a unit (i.e., a constant); hence \( p(x) \) is a constant, so \( f(ax+b) \) has no non-trivial factorization. The converse is proved similarly.