Some brief solutions for practice problems,
Section 10.3

10.3.2 This is true. Let \( I \) be an ideal of a ring \( R \), and assume that \( I \) contains a unit \( u \). Let \( s = u^{-1} \); since \( I \) is an ideal, then \( I \) contains \( su = 1 \). Since \( I \) contains 1, then \( I \) contains \( ri = r \) for all \( r \in R \). So \( I = R = \{1\} \).

10.3.6 No. This set contains the constant polynomial 2 but not 2x, so it is not closed under multiplication by elements of the ring.

10.3.7 Given a nonzero ideal \( I \) and a nonzero Gaussian integer \( z = a + bi \in I \), then \( \overline{z} = a^2 + b^2 \) is a nonzero integer in \( I \).

10.3.9 The naïve description is that the kernel is all polynomials \( f(x) \) in \( \mathbb{Z}[x] \) so that \( f(1 + \sqrt{2}) = 0 \). The more precise description is that it is all polynomials which are divisible by \( x^2 - 2x - 1 \). An analogous problem is this: if you know that \( f(x) \) is a polynomial with real coefficients, and if you know that the complex number \( a + bi \) is a root of \( f(x) \), then you can conclude that the complex number \( a - bi \) is also a root of \( f(x) \). In this case, if \( f(x) \) is a polynomial with integer coefficients, and if \( 1 + \sqrt{2} \) is a root, then so is \( 1 - \sqrt{2} \). (I’ll explain why later.) So if \( f(x) \) is in the kernel, it is divisible by both \( x - (1 + \sqrt{2}) \) and \( x - (1 - \sqrt{2}) \); thus it is divisible by

\[
(x - (1 + \sqrt{2}))(x - (1 - \sqrt{2})) = x^2 - 2x - 1.
\]

Now I have to explain why \( 1 - \sqrt{2} \) is a root, if \( 1 + \sqrt{2} \) is. In the analogous situation, if \( f(x) = \sum a_nx^n \) is a polynomial with real coefficients and \( z \) is a complex root of \( f \), take complex conjugates:

\[
\overline{0} = \overline{f(z)} = \sum \overline{a_n}z^n = \sum a_n\overline{z^n} = f(\overline{z}).
\]

(Since each \( a_n \) is real, then \( \overline{a_n} = a_n \).) In this case, define a similar operation on \( \mathbb{Z}[\sqrt{2}] \): let \( a + \sqrt{2}b = a - \sqrt{2}b \). You can check that \( \overline{w + z} = \overline{w} + \overline{z} \) and \( \overline{wz} = \overline{w} \overline{z} \). Certainly if \( a \) is in \( \mathbb{Z} \), then \( \overline{a} = a \). On the other hand, \( 1 + \sqrt{2} = 1 - \sqrt{2} \). Now repeat the complex conjugation argument to show that if \( f \) is a polynomial with integer coefficients (not with coefficients in \( \mathbb{Z}[\sqrt{2}] \)), and if \( f(1 + \sqrt{2}) = 0 \), then \( f(1 - \sqrt{2}) = 0 \).

10.3.26 The ideals in \( R[[t]] \) are \( \{0\} \) and \( (t^n) \) for each \( n \geq 0 \). These are certainly ideals, so I have to show that if \( I \) is an ideal in \( R[[t]] \), then it is equal to one of these.

Suppose that \( I \) is a nonzero ideal, and let

\[
f(t) = a_0t^n + a_{n+1}t^{n+1} + \cdots
\]

be a power series in \( I \), with \( a_n \neq 0 \). Furthermore, assume that if \( g(t) = bt^k + b_{k+1}t^{k+1} + \cdots \) is any power series in \( I \), then \( k \geq n \): every power series in \( I \) starts with a \( t^k \) term, where \( k \geq n \). In other words, assume that every power series in \( I \) is a multiple of \( t^n \). I claim that \( I = (t^n) \). The preceding assumption implies that \( I \subseteq (t^n) \). To show equality, I’ll show that \( t^n \) is a multiple of \( f(t) \); this implies that \( t^n \in I \) so \( (t^n) \subseteq I \).

Write \( f(t) = t^d(a_n + a_{n+1}t + a_{n+2}t^2 + \cdots) \). The term in parentheses is invertible, by problem 6 in section 10.2: there is a power series \( h(t) \) so that

\[
h(t)(a_n + a_{n+1}t + a_{n+2}t^2 + \cdots) = 1.
\]
Therefore $f(t)h(t) = t^n$, so $t^n \in I$, as desired.

10.3.27 The ideal $(x, y)$ generated by $x$ and $y$ is not principal. To prove this, suppose that it were generated by a single polynomial $f(x, y)$.

Then $x$ would be a multiple of $f(x, y)$, which means that there would be no $y$ terms in $f(x, y)$: if $ax^iy^j$ is the term in $f(x, y)$ with the highest power of $y$, and if $g(x, y)$ is another polynomial so that $bx^my^n$ is the term with the highest power of $y$, then in the product, there will be a nonzero term $abx^{i+m}y^{j+n}$. So if $j > 0$, $f(x, y)g(x, y)$ cannot equal $x$.

Similarly, since $y$ should be a multiple of $f(x, y)$, the highest power of $x$ present must be zero. Thus $f(x, y)$ must be just a constant polynomial: $f(x, y) = a_0$ for some nonzero $a_0 \in F$. But then the ideal generated by $f(x, y)$ is all of $F[x, y]$. Since $1 \not\in (x, y)$, this means that $(x, y) \neq (f(x, y))$.