Which polynomials $f(x) \in \mathbf{Q}[x]$ have the property that $f(n)$ is an integer for all integers n?

(a) For each integer $j \geq 0$, define $p_j(x) \in \mathbf{Q}[x]$ by

$$p_j(x) = \frac{x(x-1)(x-2)\dots(x-j+1)}{j!}.$$

(When $j = 0$, $p_0(x) = 1$.) Show that any polynomial in $\mathbf{Q}[x]$ can be written as a rational linear combination of the polynomials $p_j(x)$.

(b) Let I be this set of polynomials:

$$I = \{f(x) \in \mathbf{Q}[x] : f(n) \in \mathbf{Z} \text{ for all } n \in \mathbf{Z}\}.$$

Show that $f(x) \in I$ if and only if $f(x)$ is an integer linear combination of the polynomials $p_j(x)$. [Hint: if $f(x) \in I$, evaluate $f(x)$ at the integers 0, 1, $\dots k$.]

(This is from D'Angelo and West, *Mathematical Thinking*, 2nd edition, problem 5.65.)