**Mathematics 310A Final Exam** Name: Answers

June 6, 2005

**Instructions**: This exam is closed book, no calculators allowed. You may use one 3” × 5” sheet of notes (handwritten, and you may use both sides). In order to receive full credit, **you must justify your answers**, unless the problem states otherwise.

1. Let $k$ be an integer with $k \geq 2$.

   (a) (8 points) Use induction to prove that $k-1$ divides $k^n - 1$ for all $n \geq 1$.

   **Solution:** The base case: when $n = 1$, the statement is that $k-1$ divides $k-1$. This is certainly true, so this starts the induction.

   Now for the inductive step: assume that $k-1$ divides $k^n - 1$, and prove that $k-1$ divides $k^{n+1} - 1$. Rewrite $k^{n+1} - 1$ as

   $$k^{n+1} - 1 = k(k^n) - 1$$

   I want to get $k^n - 1$ involved, so continue rewriting:

   $$\begin{aligned} &= k(k^n - 1) + k - 1 \\ &= k(k^n - 1) + (k - 1). \end{aligned}$$

   By the inductive hypothesis, $k-1$ divides the first term. $k-1$ divides the second term by the base case. Therefore $k-1$ divides their sum, which is $k^{n+1} - 1$. This completes the inductive step, and hence the induction.

   (b) (8 points) Use modular arithmetic to prove that $k-1$ divides $k^n - 1$ for all $n \geq 1$.

   **Solution:** Since $k-1$ is divisible by $k-1$, I get the congruence $k - 1 \equiv 0 \pmod{k-1}$, or equivalently $k \equiv 1 \pmod{k-1}$. Raise both sides to the $n$th power: $k^n \equiv 1^n \pmod{k-1}$. Since $1^n = 1$, I can rewrite this as $k^n \equiv 1 \pmod{k-1}$ or $k^n - 1 \equiv 0 \pmod{k-1}$. This means that $k^n - 1$ is divisible by $k-1$.

2. (10 points) Let $p$ be a prime number. Prove that the square root of $p$ is irrational.

**Solution:** The square root of $p$ is a solution to the equation $x^2 - p = 0$. According to the Rational Zeroes Theorem, if this has a rational solution $a/b$, then $b$ must divide 1 (so $b = \pm 1$) and $a$ must divide $p$ (so $a = \pm p$ or $a = \pm 1$). That means the only possible rational solutions to this are $\pm 1, \pm p$. It is easy to check that these don't work, and therefore the equation has no rational solutions.

Alternatively, you can prove this by contradiction; just imitate the proof that $\sqrt{2}$ is irrational. (Example 8.13 in the book.)

3. (10 points) Let $n$ be a positive integer. Prove $\binom{2n}{n} = 2\binom{2n-1}{n-1}$ by counting a set in two ways. [Hint: Let $A$ be a set of size $2n$ and let $x$ be an element of $A$. Look at the $n$-element subsets of $A$ – they either contain $x$ or they don't.]

**Solution:** Following the hint, let $A$ be a set of size $2n$. Then the left side of the equation, $\binom{2n}{n}$, is the number of $n$ element subsets of $A$.

The number of $n$-element subsets of $A$ which contain $x$ is $\binom{2n-1}{n-1}$: since $x$ is in the subset, choose $n-1$ elements to get a total of $n$, and those $n-1$ should be chosen from the set $A - \{x\}$, which is a set of size $2n-1$. So there are $\binom{2n-1}{n-1}$ subsets of $A$ of size $n$ which contain $x$.

The number of $n$-element subsets of $A$ which do not contain $x$ is $\binom{2n-1}{n}$: choose $n$ elements from the set $A - \{x\}$, which has size $2n-1$. Now apply the formula $\binom{m}{k} = \binom{m}{m-k}$, with $m = 2n-1$ and $k = n$: $\binom{2n-1}{n} = \binom{2n-1}{2n-1-n} = \binom{2n-1}{n-1}$. Therefore there are $\binom{2n-1}{n-1}$ subsets of $A$ of size $n$ which do not contain $x$.

Since every subset of $A$ of size $n$ either contains $x$ or doesn't, we get the equality

$$\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n-1},$$

as desired.

4. (8 points) A standard deck of cards has four suits with thirteen cards in each suit. A flush is a five-card hand in which all of the cards are in the same suit. What is the probability that a random five-card hand is a flush? Express your answer using binomial coefficients.

**Solution:** To get a flush, first pick one of the four suits (contributing a factor of $\binom{4}{1}$), and then pick 5 of the 13 cards in that suit (contributing a factor of $\binom{13}{5}$). Divide this by the total number of five-card hands to get the probability:

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}.$$

5. For each of the following functions, tell me whether it is injective, and tell me whether it is surjective. (That is, for each of these, write "injective" or "not injective", and write "surjective" or "not surjective".) Give brief justifications for your answers.

(a) (4 points) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-x^2}$

**Solution:** Not injective: $f(2) = f(-2)$. Not surjective: $f(x) > 0$ for all $x$, so $f(x) \neq 0$ for all $x$.

(b) (4 points) $g : \mathbb{N} \to [-1, 1]$ defined by $g(n) = \sin(2\pi n)$

**Solution:** Not injective: $g(n) = 0$ for all $n$. Not surjective: $g(n)$ is never 1.

(c) (4 points) $h : \mathbb{N} \to \mathbb{N}$ defined by $h(n) = \begin{cases} 2n-1 & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

**Solution:** Not injective: $h(2) = 3 = h(5)$. Surjective: for any $k \in \mathbb{N}$, $h(2k-1) = k$, so $k$ is in the image of $h$.

6. (10 points) Let $A$ and $C$ be sets and let $f : A \to C$ and $g : C \to A$ be functions. True or false: if $f \circ g$ and $g \circ f$ are both bijective, then $f$ is bijective. If it's true, prove it. If it's false, give a counterexample.

**Solution:** The statement is true. Since $f \circ g$ is bijective, it's also surjective, so for every $y \in C$, there is an element $z \in C$ so that $(f \circ g)(z)) = y$. Therefore $f$ is surjective: for every $y \in C$, there is an element in $A$ (namely $g(z)$) which $f$ sends to $y$: $f(g(z)) = y$.

Since $g \circ f$ is bijective, it's also injective, so for $a, b \in A$ with $a \neq b$, $(g \circ f)(a) \neq (g \circ f)(b)$. That is, $g(f(a)) \neq g(f(b))$. That is, when I apply $g$ to $f(a)$, I get something different from if I apply it to $f(b)$. Therefore $f(a)$ must be different from $f(b)$: if they were equal, they would get sent to the same element by the function $g$. Summarizing, $a \neq b \Longrightarrow f(a) \neq f(b)$, which means that $f$ is injective.

Since $f$ is both surjective and injective, it's bijective.

7. Write the negation of each of the following statements without using phrases of negation (such as "it is not true that . . . "). No justifications are required for this problem.

   (a) (3 points) For all $a, x \in \mathbb{R}$, there is a unique $y$ such that $x^4 y + ay + x = 0$.

   **Solution:** For some $a, x \in \mathbb{R}$, either there are no values of $y$ or there is more than one value of $y$ such that $x^4 y + ay + x = 0$.

   (b) (3 points) For every $\varepsilon > 0$, there is a $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$.

   **Solution:** There exists $\varepsilon > 0$ so that for all $\delta > 0$, $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

   (c) (3 points) There exist natural numbers $a$ and $b$ such that $\frac{a}{b} = \sqrt{2}$.

   **Solution:** For all natural numbers $a$ and $b$, $\frac{a}{b} \neq \sqrt{2}$.