Much of this (up to and including cofinality) is taken from *Set Theory* by Thomas Jech [Jec03], but you can probably find it in any reasonable set theory book. The rest is from Hovey's book *Model Categories* [Hov99].

1 Well-ordered sets

A linearly ordered set $(P,<)$ is *well-ordered* if every nonempty subset of P has a least element. A map f of ordered sets is *increasing* if $x \geq y \Rightarrow f(x) \geq f(y)$.

Lemma 1.1. *Suppose that $(W,<)$ is well-ordered. If $f: W \to W$ is an increasing function, then $f(x) \geq x$ for all $x \in W$.*

Corollary 1.2. *The only automorphism of a well-ordered set is the identity.*

Corollary 1.3. *If W_1 and W_2 are isomorphic well-ordered sets, there is a unique isomorphism between them.*

Given a well-ordered set W and an element $u \in W$, the *initial segment* given by u is $\{x \in W : x < u\}$.

Lemma 1.4. *No well-ordered set is isomorphic to an initial segment of itself.*

Theorem 1.5. *For any well-ordered sets W_1 and W_2, exactly one of the following holds:*

1. *W_1 is isomorphic to W_2.*
2. *W_1 is isomorphic to an initial segment of W_2.*
3. *W_2 is isomorphic to an initial segment of W_1.*

If W_1 and W_2 are isomorphic well-ordered sets, say that they have the same *order type*.

2 Ordinals

A set T is *transitive* if every element of T is a subset of T. (That is, T is a subset of its power set.) A set is an *ordinal number* (or an *ordinal*) if it is transitive and well-ordered by $\in$. For ordinals α and β, define $\alpha < \beta$ if $\alpha \in \beta$.

- $0 = \emptyset$ is an ordinal.
- If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.
- If α and β are distinct ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$.
- If α and β are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$.

The class of ordinals, Ord, has these properties:

- Ord is linearly ordered by $<$.

- For each $\alpha \in \text{Ord}$, $\alpha = \{\beta : \beta < \alpha\}$.
- For each α, $\alpha \cup \{\alpha\}$ is an ordinal, and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$.

So define $\alpha + 1$ to be $\alpha \cup \{\alpha\}$; this is called the *successor* to α. If $\alpha = \beta + 1$ for some β, then α is called a *successor ordinal*. If α is not a successor ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\}$, and α is called a *limit ordinal*.

Theorem 2.1. *Every well-ordered set is isomorphic to a unique ordinal number.*

Let β be a limit ordinal. A β-*sequence* is a function whose domain is β:

$$\langle a_\xi : \xi < \beta \rangle.$$

If $\langle \alpha_\xi : \xi < \beta \rangle$ is a *nondecreasing* sequence of ordinals (so $\xi < \eta$ implies $\alpha_\xi < \alpha_\eta$), then the *limit* of the sequence is

$$\lim_{\xi \to \beta} \alpha_\xi = \sup\{\alpha_\xi : \xi < \beta\}.$$

Examples of ordinals:

$$\begin{aligned} 0 &= \{\} \\ 1 &= \{0\} \\ 2 &= \{0,1\} \\ 3 &= \{0,1,2\} \\ &\vdots \\ \omega &= \{0,1,2,\dots\} \\ \omega + 1 &= \{0,1,2,\dots;\omega\} \\ &\vdots \end{aligned}$$

3 Cardinals

An ordinal α is a *cardinal number* if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$.

If W is well-ordered, there is an ordinal α such that $|W| = |\alpha|$, so let $|W|$ denote the least ordinal α such that $|W| = |\alpha|$. (By the axiom of choice, every set can be well-ordered, so we can extend this notation to any set W.)

Note: all infinite cardinals are limit ordinals.

4 Cofinality and κ-filtered ordinals

Let $\alpha > 0$ be a limit ordinal. An increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, for β a limit ordinal, is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subset \alpha$ is *cofinal* in α if $\sup A = \alpha$.

For α an infinite limit ordinal, let $\operatorname{cf}\alpha$ be the least limit ordinal β such that there is an increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$ with $\lim_{\xi \to \beta} \alpha_\xi = \alpha$; this is called the *cofinality* of α. Note that $\operatorname{cf}\alpha \leq \alpha$ always.

From [Hov99, Section 2.1]: given a cardinal κ, an ordinal α is κ-*filtered* if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leq \kappa$, then $\sup A < \alpha$. This holds if and only if the cofinality of α is greater than κ.

For example, if κ is finite, then a κ-filtered ordinal is just an infinite limit ordinal. If κ is infinite, the smallest κ-filtered ordinal is the first cardinal κ_1 larger than κ.

5 Smallness

(This material is taken from [Hov99, Section 2.1].)

Suppose that $\mathcal{C}$ is a category with all small colimits, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, A is an object of $\mathcal{C}$ and κ is a cardinal. We say that A is κ-*small relative to* $\mathcal{D}$ if, for all κ-filtered ordinals λ and all λ-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots,$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$, the map of sets

$$\operatorname{colim}_{\beta<\lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \operatorname{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. We say that A is *small relative to* $\mathcal{D}$ if it is κ-small for some κ. We say that A is *small* if it is small relative to the whole category $\mathcal{C}$. We say that A is *finite relative to* $\mathcal{D}$ if it is κ-small relative to $\mathcal{D}$ for some finite cardinal κ, and similarly, A is *finite* if it is finite relative to $\mathcal{C}$. Finite means that maps from A commute with colimits of arbitrary λ-sequences, for any limit ordinal λ.

Examples:

- In the category of sets, every set is small. The finite sets (in the usual sense) are precisely the sets that are finite according to the above definition.
- In the category of R-modules, every module is small. Every finitely presented R-module is finite.
- In the category of topological spaces, the set $\{0, 1\}$ with the indiscrete topology is not small.

References

[Hov99] M. Hovey, *Model categories*, American Mathematical Society, Providence, RI, 1999.

[Jec03] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.