Much of this (up to and including cofinality) is taken from Set Theory by Thomas Jech [Jec03], but you can probably find it in any reasonable set theory book. The rest is from Hovey’s book Model Categories [Hov99].

1 Well-ordered sets

A linearly ordered set \((P, <)\) is well-ordered if every nonempty subset of \(P\) has a least element. A map \(f\) of ordered sets is increasing if \(x \geq y \Rightarrow f(x) \geq f(y)\).

Lemma 1.1. Suppose that \((W, <)\) is well-ordered. If \(f : W \to W\) is an increasing function, then \(f(x) \geq x\) for all \(x \in W\).

Corollary 1.2. The only automorphism of a well-ordered set is the identity.

Corollary 1.3. If \(W_1\) and \(W_2\) are isomorphic well-ordered sets, there is a unique isomorphism between them.

Given a well-ordered set \(W\) and an element \(u \in W\), the initial segment given by \(u\) is \(\{x \in W : x < u\}\).

Lemma 1.4. No well-ordered set is isomorphic to an initial segment of itself.

Theorem 1.5. For any well-ordered sets \(W_1\) and \(W_2\), exactly one of the following holds:

1. \(W_1\) is isomorphic to \(W_2\).
2. \(W_1\) is isomorphic to an initial segment of \(W_2\).
3. \(W_2\) is isomorphic to an initial segment of \(W_1\).

If \(W_1\) and \(W_2\) are isomorphic well-ordered sets, say that they have the same order type.

2 Ordinals

A set \(T\) is transitive if every element of \(T\) is a subset of \(T\). (That is, \(T\) is a subset of its power set.) A set is an ordinal number (or an ordinal) if it is transitive and well-ordered by \(\in\). For ordinals \(\alpha\) and \(\beta\), define \(\alpha < \beta\) if \(\alpha \in \beta\).

- \(0 = \emptyset\) is an ordinal.
- If \(\alpha\) is an ordinal and \(\beta \in \alpha\), then \(\beta\) is an ordinal.
- If \(\alpha\) and \(\beta\) are distinct ordinals and \(\alpha \subset \beta\), then \(\alpha \in \beta\).
- If \(\alpha\) and \(\beta\) are ordinals, then either \(\alpha \subset \beta\) or \(\beta \subset \alpha\).

The class of ordinals, \(\text{Ord}\), has these properties:

- \(\text{Ord}\) is linearly ordered by \(<\).
• For each $\alpha \in \text{Ord}$, $\alpha = \{ \beta : \beta < \alpha \}$.

• For each $\alpha$, $\alpha \cup \{ \alpha \}$ is an ordinal, and $\alpha \cup \{ \alpha \} = \inf \{ \beta : \beta > \alpha \}$.

So define $\alpha + 1$ to be $\alpha \cup \{ \alpha \}$; this is called the successor to $\alpha$. If $\alpha = \beta + 1$ for some $\beta$, then $\alpha$ is called a successor ordinal. If $\alpha$ is not a successor ordinal, then $\alpha = \sup \{ \beta : \beta < \alpha \}$, and $\alpha$ is called a limit ordinal.

**Theorem 2.1.** Every well-ordered set is isomorphic to a unique ordinal number.

Let $\beta$ be a limit ordinal. A $\beta$-sequence is a function whose domain is $\beta$:

$$(\alpha_\xi : \xi < \beta).$$

If $\langle \alpha_\xi : \xi < \beta \rangle$ is a nondecreasing sequence of ordinals (so $\xi < \eta$ implies $\alpha_\xi < \alpha_\eta$), then the limit of the sequence is

$$\lim_{\xi \to \beta} \alpha_\xi = \sup \{ \alpha_\xi : \xi < \beta \}.$$ 

**Examples of ordinals:**

\begin{align*}
0 &= \{} \\
1 &= \{ 0 \} \\
2 &= \{ 0, 1 \} \\
3 &= \{ 0, 1, 2 \} \\
\vdots \\
\omega &= \{ 0, 1, 2, \ldots \} \\
\omega + 1 &= \{ 0, 1, 2, \ldots ; \omega \} \\
\vdots
\end{align*}

### 3 Cardinals

An ordinal $\alpha$ is a cardinal number if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$.

If $W$ is well-ordered, there is an ordinal $\alpha$ such that $|W| = |\alpha|$, so let $|W|$ denote the least ordinal $\alpha$ such that $|W| = |\alpha|$. (By the axiom of choice, every set can be well-ordered, so we can extend this notation to any set $W$.)

Note: all infinite cardinals are limit ordinals.

### 4 Cofinality and $\kappa$-filtered ordinals

Let $\alpha > 0$ be a limit ordinal. An increasing $\beta$-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, for $\beta$ a limit ordinal, is cofinal in $\alpha$ if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subset \alpha$ is cofinal in $\alpha$ if $\sup A = \alpha$. 

2
For $\alpha$ an infinite limit ordinal, let $\text{cf} \alpha$ be the least limit ordinal $\beta$ such that there is an increasing $\beta$-sequence $\langle \alpha_\xi : \xi < \beta \rangle$ with $\lim_{\xi \to \beta} \alpha_\xi = \alpha$; this is called the cofinality of $\alpha$. Note that $\text{cf} \alpha \leq \alpha$ always.

From [Hov99, Section 2.1]: given a cardinal $\kappa$, an ordinal $\alpha$ is $\kappa$-filtered if it is a limit ordinal and, if $A \subseteq \alpha$ and $|A| \leq \kappa$, then $\sup A < \alpha$. This holds if and only if the cofinality of $\alpha$ is greater than $\kappa$.

For example, if $\kappa$ is finite, then a $\kappa$-filtered ordinal is just an infinite limit ordinal. If $\kappa$ is infinite, the smallest $\kappa$-filtered ordinal is the first cardinal $\kappa_1$ larger than $\kappa$.

5 Smallness

(This material is taken from [Hov99, Section 2.1].)

Suppose that $\mathcal{C}$ is a category with all small colimits, $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$, $A$ is an object of $\mathcal{C}$ and $\kappa$ is a cardinal. We say that $A$ is $\kappa$-small relative to $\mathcal{D}$ if, for all $\kappa$-filtered ordinals $\lambda$ and all $\lambda$-sequences

$$X_0 \to X_1 \to \cdots \to X_\beta \to \cdots,$$

such that each map $X_\beta \to X_{\beta+1}$ is in $\mathcal{D}$, the map of sets

$$\text{colim}_{\beta < \lambda} \mathcal{C}(A, X_\beta) \to \mathcal{C}(A, \text{colim}_{\beta < \lambda} X_\beta)$$

is an isomorphism. We say that $A$ is small relative to $\mathcal{D}$ if it is $\kappa$-small for some $\kappa$. We say that $A$ is small if it is small relative to the whole category $\mathcal{C}$. We say that $A$ is finite relative to $\mathcal{D}$ if it is $\kappa$-small relative to $\mathcal{D}$ for some finite cardinal $\kappa$, and similarly, $A$ is finite if it is finite relative to $\mathcal{C}$. Finite means that maps from $A$ commute with colimits of arbitrary $\lambda$-sequences, for any limit ordinal $\lambda$.

Examples:

- In the category of sets, every set is small. The finite sets (in the usual sense) are precisely the sets that are finite according to the above definition.
- In the category of $R$-modules, every module is small. Every finitely presented $R$-module is finite.
- In the category of topological spaces, the set $\{0, 1\}$ with the indiscrete topology is not small.

References
