# Mathematics 506 Spring 2004

**Instructions**: You may look at any notes and books when doing these problems. In the solutions, you may use any result from the reading or from lecture; do not use any results that we have not proved in class or in the reading. If you use other books (for example, if you find the solution to one of these problems in some book, or if you base your solution on an idea found in another book), you must provide a reference.

You may ask John Palmieri questions, but do not discuss the exam with anyone else.

**Due: noon, Wednesday, June 9.**

1. Let $k$ be a field. Prove that the ring $k[[x]]$ of formal power series with coefficients in $k$ is Noetherian.

2. Let $k$ be an algebraically closed field.
   (a) Let $Y$ be the plane curve $y = x^2$ over $k$ – that is, $Y = \{(x,y) \in \mathbf{A}^2 : y - x^2 = 0\}$. Show that its coordinate ring $k[Y] = k[\mathbf{A}^2]/I(Y)$ is isomorphic to a polynomial ring in one variable over $k$.
   (b) Let $Z$ be the plane curve $xy = 1$. Show that $k[Z]$ is not isomorphic to a polynomial ring in one variable over $k$. (Therefore $Y$ and $Z$ are not isomorphic as algebraic sets.)

3. Let $G$ be a finite group, and let $A$ be an abelian group with trivial $G$-action. Indeed, assume that all groups which appear in this problem as coefficients have a trivial $G$-action.
   (a) Show that $H^0(G,A) \cong A$.
   (b) Show that $H^1(G,A) \cong \mathrm{Hom}_{\mathrm{groups}}(G,A)$, the set of group homomorphisms from $G$ to $A$. Hence if $\mathbf{Z}$ is the integers and $\mathbf{C}$ is the complex numbers, then $H^1(G,\mathbf{Z}) = 0 = H^1(G,\mathbf{C})$.
   (c) Show that $H^2(G,\mathbf{C}) = 0$.
   (d) Use the short exact sequence
   $$0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}/\mathbf{Z} \to 0$$
   to conclude that $H^2(G,\mathbf{Z}) \cong \mathrm{Hom}_{\mathrm{groups}}(G,\mathbf{C}^\times)$, the set of one-dimensional complex representations of $G$.

4. Let $G$ be a finite group, $k$ an algebraically closed field, and $V$ an irreducible $k$-linear representation of $G$. Show that $\mathrm{Hom}_{kG}(V,V)$ is isomorphic to $k$, as rings.

5. Show that every element of a finite group $G$ is conjugate to its inverse if and only if every character on $G$ is real-valued. (Here, by "character" I mean complex character, and "real-valued" means that $\chi(g)$ is real for every character $\chi$ and for every $g \in G$.)