Mathematics 506 Spring 2004

1. Let k be a field. Prove that the ring $k[[x]]$ of formal power series with coefficients in k is Noetherian.

   Solution. (10 points) First, note that you can prove the more general statement that $R[[x]]$ is Noetherian if R is, but the argument simplifies considerably if you use the fact that you're working with field coefficients. So I decided not to give full credit if you ignored the fact that you were working with a field.

   Let I be an ideal in $k[[x]]$. I want to show that I is finitely generated. We may assume that I is nonzero. Let d be the largest number so that x^d divides every element of I; equivalently, if I define the *codegree* of a power series $f(x) = \sum a_i x^i$ to be the index of the first nonzero coefficient, then d is the minimum of the codegrees of the elements of I.

   Since x^d divides every element of I, $I \subseteq (x^d)$. I claim that these ideals are actually equal. By assumption, there is a power series $f(x)$ in I with codegree d: there is a power series

   $$f(x) = a_d x^d + a_{d+1} x^{d+1} + a_{d+2} x^{d+2} + \cdots$$

   in I for some elements $a_i \in k$, where $a_d \neq 0$.

   Given any power series in (x^d), which is to say any power series with codegree at least d, I want to show that it is in I. I will show that it is a multiple of $f(x)$. So fix

   $$g(x) = c_n x^n + c_{n+1} x^{n+1} + c_{n+2} x_{n+2} + \cdots$$

   where $c_i \in k$ and $n \geq d$. I want to find a power series

   $$h(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$

   so that $g(x) = f(x)h(x)$. Comparing coefficients in this purported equality, I want to be able to find elements $b_i \in k$ so that for each $m \geq 0$,

   $$c_m = \sum_{\substack{i+j=m \\ i \geq d}} a_i b_j = a_d b_{m-d} + a_{d+1} b_{m-d-1} + \cdots + a_{m-1} b_1 + a_m b_0.$$

   I will find b_i inductively. Let $b_0 = 0 = b_1 = \cdots = b_{n-d-1}$. Then for $m < n$,

   $$c_m = 0 = \sum_{\substack{i+j=m \\ i \geq d}} a_i b_j.$$

   Let $b_{n-d} = a_d^{-1} c_n$. Then $c_n = a_d b_{n-d} = \sum a_i b_j$. Now assume that I have found coefficients b_i for $i < m-d$ giving the above equality for $c_0, \ldots, c_{m-1}$. Given $b_0, \ldots, b_{m-d-1}$, can I find b_{m-d} so that

   $$c_m = a_d b_{m-d} + a_{d+1} b_{m-d-1} + \cdots + a_{m-1} b_1 + a_m b_0?$$

   Yes, just solve this equation for b_{m-d} – this is possible because a_d is nonzero, and hence a unit.

   This shows that $I = (x^d)$, and so is finitely generated.

2. Let k be an algebraically closed field.

(a) Let Y be the plane curve $y = x^2$ over k – that is, $Y = \{(x,y) \in \mathbf{A}^2 : y - x^2 = 0\}$. Show that its coordinate ring $k[Y] = k[\mathbf{A}^2]/I(Y)$ is isomorphic to a polynomial ring in one variable over k.

Solution. (5 points) First, I'll point out that both parts of this problem come from Hartshorne's *Algebraic Geometry* – it's the first exercise in section I.1.

I think the quickest way to do this one is as follows: define functions $f : \mathbf{A}^1 \to Y$ and $g : Y \to \mathbf{A}^1$ by $f(a) = (a, a^2)$ and $g(x,y) = x$. These are both defined by polynomials, and hence are morphisms. They are also inverses to each other, and so define an isomorphism of algebraic sets, and thus induce an isomorphism on coordinate rings: $k[\mathbf{A}^1] = k[z] \xrightarrow{\cong} k[Y]$.

Alternatively, one can show that $I(Y) = (y - x^2)$, and use this to show directly that $k[Y] \cong k[x]$. Let us assume for now that $I(Y)$ is equal to $(y - x^2)$. Then the coordinate ring is $k[Y] = k[x,y]/(y - x^2)$. Define a map $\phi : k[x,y] \to k[z]$ by $\phi(x) = z^2$, $\phi(y) = z$. This map is onto and I claim it has kernel $(y - x^2)$. Clearly $y - x^2$ is in the kernel, and hence $(y - x^2) \subseteq \ker\phi$. On the other hand, given $f \in \ker\phi$, then f vanishes at all points of the zero set of the ideal $(y - x^2)$, so by the Nullstellensatz, some power of f is in $(y - x^2)$. Since $k[x,y]$ is a unique factorization domain, this means that $y - x^2$ divides f. Thus ϕ induces an isomorphism

$$k[x,y]/(y - x^2) \to k[z].$$

Now, to show that $I(Y) = (y - x^2)$, if $f(x,y) \in I(Y)$, then $f(x, x^2) = 0$, so f is in the kernel of the map ϕ, and so f is in $(y - x^2)$. Clearly $(y - x^2) \subseteq I(Y)$.

(b) Let Z be the plane curve $xy = 1$. Show that $k[Z]$ is not isomorphic to a polynomial ring in one variable over k. (Therefore Y and Z are not isomorphic as algebraic sets.)

Solution. (5 points) Note first that $(xy - 1) \subseteq I(Z)$, and so in the ring $k[Z] = k[x,y]/I(Z)$, (the residue classes of) x and y are both units. Also, any ring map

$$\psi : k[Z] \to k[t]$$

is determined by where it sends x and y, and these elements must go to units in $k[t]$, which means that they must go to constant polynomials (by an easy degree argument, or just see Proposition 4 in section 7.2). Thus any ring map ψ will send both x and y to constants, and so be neither onto nor one-to-one.

(One can show, without too much work, that $I(Z)$ is actually equal to $(xy - 1)$, and also that $k[x,y]/(xy - 1) \cong k[x, x^{-1}]$, but you don't need to do that for this problem.)

3. Let G be a finite group, and let A be an abelian group with trivial G-action. Indeed, assume that all groups which appear in this problem as coefficients have a trivial G-action.

   (a) Show that $H^0(G,A) \cong A$.

   Solution. (1 point) This follows from Example (1) on p. 766 of the book: if G acts trivially on A, then $A^G = A$, and by the example, $H^0(G,A) = A^G$.

   (b) Show that $H^1(G,A) \cong \mathrm{Hom}_{\text{groups}}(G,A)$, the set of group homomorphisms from G to A. Hence if $\mathbf{Z}$ is the integers and $\mathbf{C}$ is the complex numbers, then $H^1(G,\mathbf{Z}) = 0 = H^1(G,\mathbf{C})$.

   Solution. (4 points) This is actually Proposition 30 in section 17.3, which we didn't do in class, and also wasn't assigned in the reading. We can solve this problem by using the cochain complex $C^n = C^n(G,A)$ defined in the book: $C^0 = A$, $C^1 = \mathrm{Map}(G,A)$ is the set of all functions from G to A (not just the group homomorphisms), and $C^2 = \mathrm{Map}(G\times G,A)$ is the set of all functions from $G\times G$ to A. The boundary maps are given in equation (17.18):

$$\begin{aligned} d_0 : A &\longrightarrow \mathrm{Map}(G,A) \\ a &\longmapsto g\cdot a - a \end{aligned}$$

   and since A is a trivial G-module, $g \cdot a = a$, and so $d_0 = 0$: $d_0(a)$ is the map $G \to A$ sending every group element to 0.

$$\begin{aligned} d_1 : \mathrm{Map}(G,A) &\longrightarrow \mathrm{Map}(G\times G,A) \\ f &\longmapsto \big((g_1,g_2) \mapsto g_1 \cdot f(g_2) - f(g_1g_2) + f(g_1)\big), \end{aligned}$$

   and since A is a trivial module, f gets sent to the map sending $(g_1,g_2) \in G\times G$ to

$$f(g_1) + f(g_2) - f(g_1g_2).$$

   The group of cocycles in degree 1 is the kernel of d_1, which is to say all functions $f : G \to A$ satisfying

$$f(g_1) + f(g_2) - f(g_1g_2) = 0$$

   for all $g_1, g_2 \in G$. That is, the group of cocycles is precisely the set of group homomorphisms from G to A. The group of coboundaries is the zero group, and so the cohomology is as desired, $\mathrm{Hom}_{\text{groups}}(G,A)$.

   If A is an abelian group in which no nonzero element has finite order, then the only group homomorphism from G to A is the zero map. This is the case when $A = \mathbf{Z}$ or when $A = \mathbf{C}$.

   (c) Show that $H^2(G,\mathbf{C}) = 0$.

   Solution. (2 points.) There are several possible approaches. According to a result in the book, if $|G| = m$, then $mH^n(G,\mathbf{C})$ for all $n \geq 1$. I claim that $H^n(G,\mathbf{C})$ is a complex vector space for every $n \geq 0$, which would then imply that $H^n(G,\mathbf{C}) = 0$ for all $n \geq 1$. An outline for verifying this claim: the groups $C^n(G,A)$ used to compute group cohomology get their group structure entirely from A. The same goes for showing that the boundary maps are group homomorphisms. In this case, scalar multiplication on $\mathbf{C}$ makes $C^n(G,\mathbf{C})$ into a complex vector space, and the boundary maps respect this, and so are linear transformations. Thus the kernels and images of the boundary maps are vector spaces, and thus the cohomology groups are.

Another approach: I claim that for any $n \geq 0$, $H^n(G, \mathbf{C}) = \mathrm{Ext}^n_{\mathbf{Z}G}(\mathbf{Z}, \mathbf{C})$ is isomorphic to $\mathrm{Ext}^n_{\mathbf{C}G}(\mathbf{C}, \mathbf{C})$. The latter group is zero when $n > 0$, because of Maschke's theorem: since the characteristic of $\mathbf{C}$ does not divide the order of G, every $\mathbf{C}G$-module is injective, and so Ext^n is zero when $n > 0$.

To verify the claim: let

$$\cdots \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

be a resolution of $\mathbf{Z}$ by free $\mathbf{Z}G$-modules: for each i, P_i is isomorphic to a direct sum of copies of $\mathbf{Z}G$. (The authors construct such a resolution in section 17.2, and I also described how to do it in class.) Then $H^n(G, \mathbf{C})$ is the nth cohomology group of the cochain complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathbf{Z}G}(P_1, \mathbf{C}) \leftarrow \mathrm{Hom}_{\mathbf{Z}G}(P_0, \mathbf{C}) \leftarrow 0.$$

Note, by the way, that if P_i is freely generated by a set S, then $\mathrm{Hom}_{\mathbf{Z}G}(P_i, \mathbf{C}) \cong \mathrm{Hom}_{\mathrm{sets}}(S, \mathbf{C})$. Now, $\mathbf{C}$ is flat as a $\mathbf{Z}$-module, so applying $\mathbf{C} \otimes_{\mathbf{Z}} -$ to the above resolution gives an exact sequence

$$\cdots \to \mathbf{C} \otimes P_1 \to \mathbf{C} \otimes P_0 \to \mathbf{C} \otimes \mathbf{Z} \to 0.$$

Furthermore, since each P_i is a direct sum of copies of $\mathbf{Z}G$, $\mathbf{C} \otimes P_i$ is a direct sum of copies of $\mathbf{C}G$. Note also that $\mathbf{C} \otimes \mathbf{Z} \cong \mathbf{C}$, so this is a resolution of $\mathbf{C}$ by free $\mathbf{C}G$-modules. Thus $\mathrm{Ext}^n_{\mathbf{C}G}(\mathbf{C}, \mathbf{C})$ is the nth cohomology group of the cochain complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathbf{C}G}(\mathbf{C} \otimes P_1, \mathbf{C}) \leftarrow \mathrm{Hom}_{\mathbf{C}G}(\mathbf{C} \otimes P_0, \mathbf{C}) \leftarrow 0.$$

Note that if P_i is freely generated as a $\mathbf{Z}G$-module by a set S, then $\mathbf{C} \otimes P_i$ is freely generated as a $\mathbf{C}G$-module by the same set, and so

$$\mathrm{Hom}_{\mathbf{Z}G}(P_i, \mathbf{C}) \cong \mathrm{Hom}_{\mathrm{sets}}(S, \mathbf{C}) = \mathrm{Hom}_{\mathbf{C}G}(\mathbf{C} \otimes P_i, \mathbf{C}).$$

So the groups making up these two cochain complexes are the same. One can also verify that the maps are the same, and thus their cohomology groups are the same, as claimed.

(d) Use the short exact sequence

$$0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}/\mathbf{Z} \to 0$$

to conclude that $H^2(G, \mathbf{Z}) \cong \mathrm{Hom}_{\mathrm{groups}}(G, \mathbf{C}^{\times})$, the set of one-dimensional complex representations of G.

Solution. (3 points) This problem was modeled on exercise 6.1.6 in Weibel, *An Introduction to Homological Algebra*.

The short exact sequence yields a long exact sequence in cohomology, a portion of which is

$$H^1(G, \mathbf{C}) \to H^1(G, \mathbf{C}/\mathbf{Z}) \to H^2(G, \mathbf{Z}) \to H^2(G, \mathbf{C}).$$

By the previous two parts, $H^1(G,\mathbf{C}) = 0 = H^2(G,\mathbf{C})$, so by exactness,

$$H^1(G,\mathbf{C}/\mathbf{Z}) \cong H^2(G,\mathbf{Z}).$$

By part (b), I can rewrite the left side as

$$H^1(G,\mathbf{C}/\mathbf{Z}) \cong \mathrm{Hom}_{\mathrm{groups}}(G,\mathbf{C}/\mathbf{Z}),$$

so it suffices to show that there is a group isomorphism $\mathbf{C}/\mathbf{Z} \cong \mathbf{C}^\times$. Define a map $\phi : \mathbf{C} \to \mathbf{C}^\times$ by $\phi(z) = e^{2\pi i z}$. This is a group map, it's onto, and its kernel is the subgroup of integers in $\mathbf{C}$.

4. Let G be a finite group, k an algebraically closed field, and V an irreducible k-linear representation of G. Show that $\mathrm{Hom}_{kG}(V,V)$ is isomorphic to k, as rings.

   Solution. (5 points) (This problem is Dummit & Foote, problem 16 in section 18.2.)

   Since V is a simple kG-module, then by Schur's lemma (Lemma 7 in section 18.2), $\mathrm{Hom}_{kG}(V,V)$ is a division ring; I'll call it Δ. I would like to use Proposition 9 in section 18.2 to conclude that Δ is isomorphic to k, and to do this, I have to show that Δ is finite-dimensional as a k-vector space, and that k is contained in the center of Δ.

   First of all, since V is irreducible, then V must be finite-dimensional: given any $x \in V$, there is a kG-module map

   $$kG \to V$$

   sending 1 to x. Since kG is finite-dimensional, the image of this map is a finite-dimensional submodule containing x. Since V is irreducible, if x is nonzero, this submodule must be all of x, and thus V is finite-dimensional. (Indeed, the dimension of V is at most the dimension of kG.)

   So the space of linear transformations $\mathrm{Hom}_k(V,V)$ is finite-dimensional, and $\mathrm{Hom}_{kG}(V,V)$ is a sub-vector space of it, and so is also finite-dimensional.

   Next, for any $\alpha \in k$, the linear transformation determined by the matrix αI (α times the identity matrix) is the same as scalar multiplication by α, and so is central in Δ. This map is also a kG-module map: for any $g \in G$ and $v \in V$, $g(\alpha v) = \alpha(gv)$. Thus the map $k \to \Delta$, sending α to αI, maps k injectively to the center of Δ.

   Now apply Proposition 9 of section 18.2 to yield the result.

   Alternatively, argue as above to show that V is finite-dimensional. For any kG-module map $f : V \to V$, view f as a linear transformation and find an eigenvalue α; then $f - \alpha I$ is in $\mathrm{Hom}_{kG}(V,V)$, and is singular (since α is an eigenvalue). Thus it must be zero, since $\mathrm{Hom}_{kG}(V,V)$ is a division ring, so $f = \alpha I$. This is true for any $f \in \mathrm{Hom}_{kG}(V,V)$; thus $\mathrm{Hom}_{kG}(V,V) \cong k$.

5. Show that every element of a finite group G is conjugate to its inverse if and only if every character on G is real-valued. (Here, by "character" I mean complex character, and "real-valued" means that $\chi(g)$ is real for every character χ and for every $g \in G$.)

   Solution. (5 points) (This problem was taken from an as yet unpublished book by Peter Webb, from the University of Minnesota.)

   My main tool will be Proposition 14 in section 18.3: for any character χ of G and any $g \in G$, $\chi(g^{-1}) = \overline{\chi(g)}$.

   If every character is real-valued, then for any character χ and any $g \in G$, $\chi(g^{-1}) = \chi(g)$, by Proposition 14. Since the irreducible characters form a basis for the class functions (statement (18.10)), then for every class function f, $f(g) = f(g^{-1})$. If g and g^{-1} were not conjugate, I could define a class function which had the value 1 on g and the value 0 on g^{-1}; since there is no such class function, g must be conjugate to g^{-1}. (Alternatively, you can use the second orthogonality relation for this part.)

   Conversely, if g is conjugate to g^{-1} for all g, then for any character χ, $\chi(g) = \chi(g^{-1})$ since χ is a class function, and on the other hand $\overline{\chi(g)} = \chi(g^{-1})$ by Proposition 14. Combining these, I find that $\chi(g) = \overline{\chi(g)}$ for all $g \in G$, which means that $\chi(g)$ is real for all $g \in G$.