Mathematics 505 Winter 2004

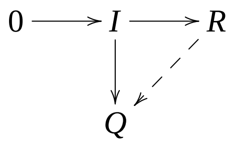
Instructions. For problems 1 and 2, you can use any result from the book or the homework. For problem 3, you can use any result except for the homework problems which establish an isomorphism between $\mathrm{Hom}_R(R,M)$ and M.

A useful fact for several of the problems: if A is a cyclic R-module generated by a, then any R-module map $A \to B$ is determined by where a goes.

1. If n and d are integers with d dividing n, then multiplication makes $\mathbf{Z}/d$ into a module over the ring $\mathbf{Z}/n$. (You don't need to prove that.)

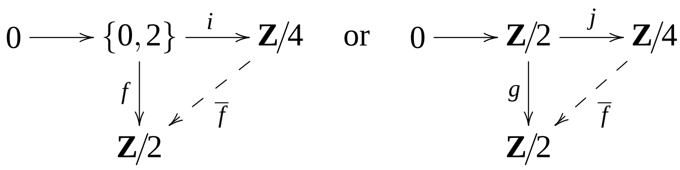
   Is $\mathbf{Z}/2$ injective as a module over $\mathbf{Z}/4$? Is $\mathbf{Z}/2$ injective as a module over $\mathbf{Z}/6$?

   <u>Solution.</u> I'll use Baer's criterion for exactness: a module Q is injective if and only if for every ideal I in R and every R-module map $I \to Q$, the dashed map in the following diagram exists, making the diagram commute:



   Note that such a map always exists if $I = (0)$ or if $I = R$, so it suffices to check with nontrivial ideals.

   $\mathbf{Z}/4$ has only one nontrivial ideal, $(2) = \{0,2\} \cong \mathbf{Z}/2$, so here is the relevant diagram:



   The map f is determined by where it sends 2 (by the "useful fact" mentioned above). The inclusion map i sends 2 to 2, and so the composite $\overline{f} \circ i$ sends $2 \in \{0,2\}$ to $\overline{f}(2) = 2\overline{f}(1)$, which is a multiple of 2, and hence is zero. So if the map f sends 2 to 1, there is no map $\overline{f}$ making the diagram commute. Hence $\mathbf{Z}/2$ is not injective over $\mathbf{Z}/4$.

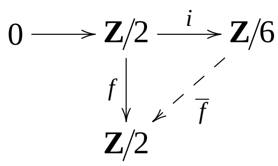
   (Equivalently, g is determined by where it sends $1 \in \mathbf{Z}/2$. j sends 1 to 2. If g sends 1 to 1, there is no map $\overline{f}$ such that $g = \overline{f} \circ j$.)

   (Alternatively, applying $\mathrm{Hom}_{\mathbf{Z}/4}(-,\mathbf{Z}/2)$ to the short exact sequence $0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0$ yields this:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(\mathbf{Z}/2,\mathbf{Z}/2) & \longrightarrow & \mathrm{Hom}(\mathbf{Z}/4,\mathbf{Z}/2) & \longrightarrow & \mathrm{Hom}(\mathbf{Z}/2,\mathbf{Z}/2) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \mathbf{Z}/2 & & \mathbf{Z}/2 & & \mathbf{Z}/2 & & \end{array}$$

This sequence cannot be exact.)

The ring $\mathbf{Z}/6$ has only two nontrivial ideals: $(2) = \{0,2,4\} \cong \mathbf{Z}/3$ and $(3) = \{0,3\} \cong \mathbf{Z}/2$. There are no nonzero maps from $\mathbf{Z}/3$ to $\mathbf{Z}/2$ (there are no nonzero group homomorphisms between these, and every module map must be a homomorphism of abelian groups), and the zero map always has an extension to the whole ring; therefore the only diagram that needs checking is this one:



The map f is determined by $f(1)$. The map i sends 1 to 3, so $\overline{f} \circ i(1) = 3\overline{f}(1) = \overline{f}(1)$. Therefore if $f(1) = k$, define $\overline{f}$ by $\overline{f}(1) = k$; this is an extension of f. Hence $\mathbf{Z}/2$ is injective over $\mathbf{Z}/6$.

By the way, this problem illustrates a general fact: $\mathbf{Z}/d$ is injective over $\mathbf{Z}/n$ if and only if $(d, \frac{n}{d}) = 1$.

2. Let k be a field and let $R = k[x]$. Show that

$$R/(x^m) \otimes_R R/(x^n) \cong R/(x^{\min(m,n)}).$$

(You may use this fact: the tensor product of two cyclic modules is cyclic.)

Solution. By the "fact," the tensor product $T = R/(x^m) \otimes_R R/(x^n)$ is cyclic, and so is isomorphic to $R/(f)$ for some polynomial f. (By a homework problem, a module is cyclic if and only if it is isomorphic to R/I for some left ideal I. Since $R[x]$ is a PID, I can write the relevant left ideal as (f) for some f.)

Let $d = \min(m,n)$. Since $x^m = 0$ and $x^n = 0$ in T, $x^d = 0$ in T. So f divides x^d, and therefore $\dim_k T \leq d$.

Define a function

$$R/(x^m) \times R/(x^n) \to R/(x^d)$$

by $(g,h) \to gh$. This is R-bilinear and surjective, and so induces a surjective R-module map $\phi : T \to R/(x^d)$. (Surjectivity of ϕ follows from surjectivity of the bilinear map, but also can be seen directly: ϕ sends the basic tensor $x^i \otimes x^j$ to x^{i+j}, and this map is pretty clearly surjective.) Since ϕ is surjective, $\dim_k T \geq d$. Thus $\dim_k T = d$, and ϕ must be an isomorphism.

By the way, this problem is a special case of a general fact: if R is a PID, then $R/(a) \otimes_R R/(b) \cong R/(a,b)$.

3. Let R be a ring with 1.

(a) Show that for any R-module M, there is an isomorphism of abelian groups $\mathrm{Hom}_R(R,M) \cong M$. (This is in fact an isomorphism of R-modules, but you don't need to show that.)

(b) Write $\alpha_M : \mathrm{Hom}_R(R,M) \to M$ and $\beta_M : M \to \mathrm{Hom}_R(R,M)$ for the maps giving the isomorphism in part (a). Given an R-module homomorphism $f : M \to N$, show that the composite

$$M \xrightarrow[\cong]{\beta_M} \mathrm{Hom}_R(R,M) \xrightarrow{\mathrm{Hom}_R(R,f)} \mathrm{Hom}_R(R,N) \xrightarrow[\cong]{\alpha_N} N$$

is equal to f.

Solution. (a) First note (by the "useful fact") that any R-module map $R \to M$ is determined by where it sends 1. Also, 1 can go to any element of M – this reflects the fact that R is a free R-module generated by 1.

Define $\alpha : \mathrm{Hom}_R(R,M) \to M$ by $\alpha(f) = f(1)$. Define $\beta : M \to \mathrm{Hom}_R(R,M)$ by making $\beta(m)$ be the map sending 1 to m (and so sending r to rm for any $r \in R$). I'll write that as $\beta(m) = (1 \mapsto m)$. Then $\alpha \circ \beta(m) = \alpha(1 \mapsto m) = m$, and $\beta \circ \alpha(f) = \beta(f(1)) = (1 \mapsto f(1))$. Since any map from R to M is determined by where 1 goes, the map $1 \mapsto f(1)$ is equal to the map f. Thus α and β are inverses to each other, and hence establish a bijection between M and $\mathrm{Hom}_R(R,M)$.

I should check that they're group homomorphisms, too:

$$\alpha(f+g) = (f+g)(1) = f(1) + g(1) = \alpha(f) + \alpha(g),$$
$$\beta(m+n) = (1 \mapsto (m+n)) = (1 \mapsto m) + (1 \mapsto n) = \beta(m) + \beta(n).$$

(b) Given $m \in M$, I'll show that applying the composite $\alpha_N \circ \mathrm{Hom}_R(R,f) \circ \beta_M$ to it yields $f(m)$:

$$\Big(\alpha_N \circ \mathrm{Hom}_R(R,f) \circ \beta_M\Big)(m) = \Big(\alpha_N \circ \mathrm{Hom}_R(R,f)\Big)(1 \mapsto m)$$

Now, the map $\mathrm{Hom}_R(R,f)$ is the map induced by f on Hom, and it just composes with f: it sends $g : R \to M$ to $f \circ g : R \to N$. So we have

$$\begin{aligned} &= \alpha_N(1 \mapsto m \mapsto f(m)) \\ &= \alpha_N(1 \mapsto f(m)) \\ &= f(m). \end{aligned}$$