Mathematics 402A Final Solutions

December 15, 2004

1. (15 points) In class I stated, but did not prove, the following classification theorem: every abelian group of order 8 is isomorphic to C_8, $C_4 \times C_2$, or $C_2 \times C_2 \times C_2$. Prove this. [Hint: imitate the classification of groups of order 6.]

   Solution. Suppose that G is an abelian group of order 8. By Lagrange's theorem, the elements of G can have order 1, 2, 4, or 8.

   If G contains an element of order 8, then G is cyclic, generated by that element: $G \approx C_8$.

   Suppose that G has no elements of order 8, but contains an element x of order 4. Let $H = \{1, x, x^2, x^3\}$ be the cyclic subgroup generated by x. If I can find an element y of order 2 which is not in H, then I claim that I'm done: given such a y, let $K = \{1, y\}$ be the cyclic subgroup generated by y. Then H and K are both normal in G (since G is abelian), $H \cap K = \{1\}$ by inspection, and also $HK = G$. To see this last equality, note that HK is a subgroup of G since H and K are normal, and it contains H and K, so its order is at least 5. Since its order must divide 8, its order must equal 8, so it is all of G. By the theorem in the book about products, I can conclude that $G \approx H \times K$. Since $H \approx C_4$ and $K \approx C_2$, I get $G \approx C_4 \times C_2$.

   So I still need to find an element in G, and not in H, of order 2. Pick $z \notin H$. If z does not have order 2, then it has order 4. In this case, z^2 has order 2. If z^2 is not in H, I'm done. If it is in H, then z^2 is either x, x^2, or x^3. If $z^2 = x$ or $z^2 = x^3$, then z has order 8, which I'm assuming doesn't happen. If $z^2 = x^2$, then zx is an element of order 2 which is not contained in H.

   Now suppose that G has no elements order 4 or 8. Let x and y be distinct elements of order 2, and let H and K be the subgroups that they each generate. Then $HK = \{1, x, y, xy\}$ is a subgroup of G and is isomorphic to $H \times K \approx C_2 \times C_2$. Find an element z not in HK; then z has order 2, and let L be the subgroup that it generates. Then HK and L are both normal, $HK \cap L = \{1\}$ by our choice of z, and $HKL = G$ – you can verify this last assertion the same way I did in the previous case. So by the theorem on products, $G \approx HK \times L \approx H \times K \times L \approx C_2 \times C_2 \times C_2$.

2. (10 points) How many rotational symmetries does a rhombicuboctahedron have?

   Solution. $\boxed{24}$

   How many rotational symmetries does a truncated tetrahedron have?

   Solution. $\boxed{12}$

   How many rotational symmetries does a cuboctahedron have?

   Solution. $\boxed{24}$

   How many rotational symmetries does a truncated cuboctahedron have?

   Solution. $\boxed{24}$

   How many rotational symmetries does a rhombicosidodecahedron have?

   Solution. $\boxed{60}$

   How many rotational symmetries does a truncated icosahedron have?

Solution. $\boxed{60}$

I'll do this one as an example. There are 12 pentagonal faces, so pick one. Its orbit consists of all 12 faces, and its stabilizer consists of the 5 rotations about an axis through the center of that face. So there are $12 \cdot 5 = 60$ total rotational symmetries. Alternatively, there are 20 hexagonal faces, so pick one. Its orbit consists of all 20 faces, but its stabilizer only contains three rotations: three of the six rotations of the hexagon send this figure to itself, but three of them don't. So there are $20 \cdot 3 = 60$ rotational symmetries.

3. (10 points) Let $C_n = \{1, x, x^2, \ldots, x^{n-1} \mid x^n = 1\}$ denote a cyclic group of order n, generated by x. What is the order of x^i, where $0 \leq i \leq n-1$? Your answer is likely to depend on i and n.

(If you can't do this in general, do special cases. For example, what if n is prime? What if n is a power of a prime? Can you answer the question for some values of i?)

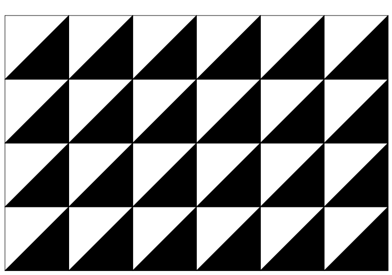
Solution. Let's see. Let m be the order of x^i. If I let d be the greatest common divisor of i and n, so that $i = dr$ and $n = ds$ with r and s relatively prime, then I claim that $m = s$.

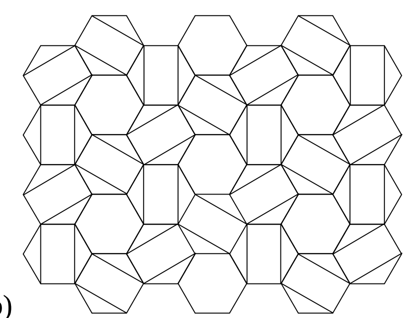
First, $x^{is} = x^{drs} = x^{rn} = 1$, so m divides s. On the other hand, since $x^{im} = 1$, then $im = drm$ is divisible by $n = ds$, which means that rm is divisible by s. Since r and s are relatively prime, this means that m is divisible by s. Since m divides s and s divides m, I can conclude that $s = m$.

Given all of this, I can rewrite the answer: the order of x^i is

$$\frac{n}{\gcd(i,n)} = \frac{\operatorname{lcm}(i,n)}{i}.$$

4. (10 points) Determine the point groups of the symmetry groups of each of these subsets of the plane. Give *brief* explanations of your answers.

(a) 

(b) 

Solution. (a) The point group contains a reflection (across a diagonal line running northwest/southeast), and no rotations: any nontrivial rotation will take a black triangle to a black triangle oriented differently – look at where the right angle goes. So the point group is isomorphic to D_1 or to C_2, generated by a single reflection.

(b) The figure has 6-fold rotational symmetry, and no reflections. So the point group is isomorphic to C_6.

5. (10 points) The following statement has some errors in it; fix the errors to produce a true statement. Explain *briefly* why the original statement was false and why the new statement is true.

   "For each integer $n \geq 0$, $GL_n(\mathbf{R})/SL_n(\mathbf{R}) \approx \mathbf{R}$."

   Solution. There are two errors. It should say

   "For each integer $n \geq \boxed{1}$, $GL_n(\mathbf{R})/SL_n(\mathbf{R}) \approx \boxed{\mathbf{R}^\times}$."

   When $n = 0$, $GL_0(\mathbf{R}) = SL_0(\mathbf{R}) = \{1\}$, the trivial group, and so the quotient group will also be isomorphic to $\{1\}$.

   (Or if you like, $GL_0(\mathbf{R})$ is not defined.)

   When $n > 0$, $SL_n(\mathbf{R})$ is the kernel of the determinant map $GL_n(\mathbf{R}) \to \mathbf{R}^\times$, so since this determinant map is surjective, the first isomorphism theorem gives an isomorphism between $GL_n(\mathbf{R})/SL_n(\mathbf{R})$ and the image, $\mathbf{R}^\times$.

6. (15 points) Consider the dihedral group D_6.

   (a) Find all of the subgroups of D_6.

   Solution. First, I'll write down the elements of D_6:

$$D_6 = \{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y \mid x^6 = 1, y^2 = 1, yx = x^5y\}.$$

   This group has order 12, so the possible orders of subgroups are 1, 2, 3, 4, 6, 12. I'll deal with these in order:

   order 1: $\{1\}$ is the only subgroup of order 1.

   order 2: each subgroup of order 2 must contain the identity element plus an element of order 2. I know the orders of the elements in D_6, so I can list all of these subgroups. The first of these is maybe the easiest to miss:

$$\{1, x^3\}, \{1, y\}, \{1, xy\}, \{1, x^2y\}, \{1, x^3y\}, \{1, x^4y\}, \{1, x^5y\}.$$

   order 3: x^2 and x^4 are the only elements of order 3, so there is only one subgroup of order 3: $\{1, x^2, x^4\}$.

   order 4: there are no elements of order 4, so any subgroup of order 4 must contain the identity element plus three elements of order 2. I know all of the elements of order 2, so the only question is, which pairs of such elements generate a subgroup of order 4? For example, y and xy do not, because their product (in one order) is $xyy = x$, so any subgroup containing y and xy must also contain x, and so must contain all of D_6. Similarly, any subgroup containing x^iy and x^jy must contain x^{i-j} and x^{j-i}. The only way for this to produce a subgroup of order 4 is if $x^{i-j} = x^3 = x^{j-1}$. So here are the subgroups of order 4:

$$\{1, x^3, y, x^3y\}, \{1, x^3, xy, x^4y\}, \{1, x^3, x^2y, x^5y\}.$$

   order 6: there is one element of order 6, namely x, so that gives me one subgroup of order 6: $\{1, x, x^2, x^3, x^4, x^5\}$. Are there any others? If there were, they would need to be isomorphic to S_3, since they wouldn't contain any elements of order 6. So they would have to contain two elements of order 3 and three elements of order 2. So they would have to contain x^2 and x^4, plus three elements of order 2. Since the resulting set must be closed, these are the only possibilities:

$$\{1,x,x^2,x^3,x^4,x^5\}, \{1,x^2,x^4,y,x^2y,x^4y\}, \{1,x^2,x^4,xy,x^3y,x^5y\}.$$

These are all subgroups.

order 12: the whole group is the only subgroup of order 12.

(b) Which ones are normal?

Solution. The trivial group $\{1\}$ and the whole group D_6 are certainly normal.

Among the subgroups of order 2, only $\{1,x^3\}$ is normal: $x(x^iy)x^{-1} = x^{i+2}y$, so $\{1,x^iy\}$ is not normal for any i.

The subgroup of order 3 is normal.

The subgroups of order 4 are not normal, by the calculation I just presented in the order 2 case.

The subgroups of order 6 are all normal, because they have index 2.

(c) What is the class equation for D_6?

Solution.. The answer is $12 = 1+1+2+2+3+3$.

In order to determine the size of the conjugacy class C_a of each element a, I'll use the formula $|D_6| = |C_a||Z(a)|$, where $Z(a)$ is the centralizer of a.

x commutes with all of the powers of x, and it doesn't commute with y, so $Z(x)$ is a proper subgroup containing at least six elements. Since the order of $Z(x)$ must divide 12, we conclude that it contains exactly six elements; thus the conjugacy class of x contains two elements. One of those elements is x; to find the other, we conjugate x by y: $yxy^{-1} = yxy = x^5y^2 = x^5$. So $C_x = \{x,x^5\}$. Similarly, the conjugacy class of x^2 is $\{x^2,x^4\}$. x^3 works a bit differently, though: x^3 does actually commute with y, so $Z(x^3)$ is all of D_6, so the conjugacy class of x^3 is $\{x^3\}$.

What is the stabilizer of y? x^i is in $Z(y)$ if and only if $x^iyx^{-i} = y$, and $x^iyx^{-i} = x^ix^iy = x^{2i}y$. This equals y if and only if $x^{2i} = 1$, which is true if and only if $i = 0$ or $i = 3$. So $Z(y)$ contains 1 and x^3. Similarly, x^iy commutes with y if and only if $i = 0$ or $i = 3$, so $Z(y) = \{1,x^3,y,x^3y\}$. Thus there are three elements conjugate to y. You can find them by conjugating y by 1, by x, and by x^2: $C_y = \{y,x^2y,x^4y\}$. Similarly, the conjugacy class of xy is $C_{xy} = \{xy,x^3y,x^5y\}$. So the class equation is

$$12 = 1+1+2+2+3+3.$$