Mathematics 412

7 February 2003

Midterm preview

Instructions: For this exam, clarity of exposition is as important as correctness of mathematics.

1. A friend comes to you and asks if a particular polynomial $p(x)$ of degree 25 in $\mathbb{F}_2[x]$ is irreducible. The friend explains that she has tried dividing $p(x)$ by every polynomial in $\mathbb{F}_2[x]$ of degree from 1 to 18 and has found that $p(x)$ is not divisible by any of them. She is getting tired of doing all these divisions and wonders if there's an easier way to check whether or not $p(x)$ is irreducible. You surprise your friend with the statement that she need not do any more work: $p(x)$ is indeed irreducible!

   Prove this; that is, use the fact that no polynomial of degree between 1 and 18 divides $p(x)$ to prove that $p(x)$ is irreducible. Do not simply quote a theorem that makes this problem trivial; rather, provide an argument "from scratch" using the given information. You may use the fact that the degree of a product of two polynomials is the sum of the degrees of the two polynomials.

2. Suppose that $f(x)$ is the cubic polynomial $x^3 - 9x + 6$ in $\mathbb{R}[x]$. Using standard graphing techniques from calculus, one can easily show that the graph of $y = f(x)$ crosses the x-axis three times. (You don't have to prove this.) This tells us that $f(x)$ has three real roots.

   (a) Use Cardano's Formula to write down an expression for one of the roots of $f(x)$ and observe that the expression you obtained is the sum of the cube roots of two non-real complex numbers.

   (b) Explain how it is possible for this expression to be a real number even though it involves non-real numbers.

3. Prove that the polynomial
$$6x^{18} - 50x^7 + 30x^2 - 15$$
   does not factor in $\mathbb{Z}[x]$ as the product $g(x)h(x)$ of two polynomials $g(x)$ and $h(x)$ whose degrees are both less than 18. (Do not simply quote and apply a major theorem. Rather, give a proof from scratch.)

4. Prove that the polynomial
$$15x^4 + 7x^3 - 4x^2 - 33$$
   does not factor in $\mathbb{Z}[x]$ as the product $g(x)h(x)$ of two polynomials $g(x)$ and $h(x)$ whose degrees are both less than 4. (You may use theorems for this problem, as long as you explain what you're using.)

5. Let K be a field.

   (a) State Bezout's Theorem for a pair of polynomials $a(x)$ and $b(x)$ in $K[x]$.

   (b) Prove the statement below.

   > Suppose that $a(x)$ and $b(x)$ are relatively prime polynomials in $K[x]$ and $a(x)$ divides the product $b(x)c(x)$ in $K[x]$. Then $a(x)$ divides $c(x)$.

   You may use Bezout's theorem in your proof. If you do, be sure to make clear where and how you are using it.