Mathematics 307I Final solutions

Part I: Laplace transforms

1. Use Laplace transforms to solve this initial value problem:

$$y'' + 2y' + 5y = 10u_\pi(t)\sin(t-\pi),$$
$$y(0) = 0,\ y'(0) = 0.$$

Solution. Apply the Laplace transform to the equation:

$$(s^2+2s+5)Y = 10e^{-\pi s}\frac{1}{s^2+1}.$$

Solve for Y and use partial fractions:

$$Y = e^{-\pi s}\frac{10}{(s^2+1)(s^2+2s+5)} = e^{-\pi s}\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+5}\right).$$

Solve for the constants. After some algebra, you should get $A = -1$, $B = 2$, $C = 1$, and $D = 0$, so

$$\begin{aligned}
Y &= e^{-\pi s}\left(\frac{-s+2}{s^2+1} + \frac{s}{s^2+2s+5}\right)\\
&= e^{-\pi s}\left(-\frac{s}{s^2+1} + 2\frac{1}{s^2+1} + \frac{s}{(s+1)^2+4}\right)\\
&= e^{-\pi s}\left(-\frac{s}{s^2+1} + 2\frac{1}{s^2+1} + \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}\right)\\
&= e^{-\pi s}\left(-\frac{s}{s^2+1} + 2\frac{1}{s^2+1} + \frac{s+1}{(s+1)^2+4} - \frac{1}{2}\frac{2}{(s+1)^2+4}\right).
\end{aligned}$$

Now take inverse Laplace transforms:

$$\boxed{y = u_\pi(t)\left(-\cos(t-\pi) + 2\sin(t-\pi) + e^{-(t-\pi)}\cos 2(t-\pi) - \frac{1}{2}e^{-(t-\pi)}\sin 2(t-\pi)\right)}.$$

If you want, you can use some easy trig identities to simplify this:

$$\boxed{y = u_\pi(t)\left(\cos t - 2\sin t + e^{\pi-t}\cos 2t - \frac{1}{2}e^{\pi-t}\sin 2t\right)}.$$

2. Use Laplace transforms to solve this initial value problem:

$$y'' - 6y' + 8y = 8,$$
$$y(0) = 0,\ y'(0) = 2.$$

Solution. Apply the Laplace transform to the equation:

$$s^2Y - 2 - 6sY + 8Y = \frac{8}{s}.$$

Solve for Y:

$$\begin{aligned} (s^2 - 6s + 8)Y - 2 &= \frac{8}{s} \\ (s^2 - 6s + 8)Y &= 2 + \frac{8}{s} = \frac{2s+8}{s} \\ Y &= \frac{2s+8}{s(s^2 - 6s + 8)} = \frac{2s+8}{s(s-2)(s-4)}. \end{aligned}$$

(This was a sticking point for many people: you didn't check to see if the quadratic in the denominator could be factored.)

Now use partial fractions:

$$Y = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-4}.$$

Solve for A, B, and C to find

$$Y = \frac{1}{s} + \frac{-3}{s-2} + \frac{2}{s-4}.$$

Apply the inverse Laplace transform to get the solution:

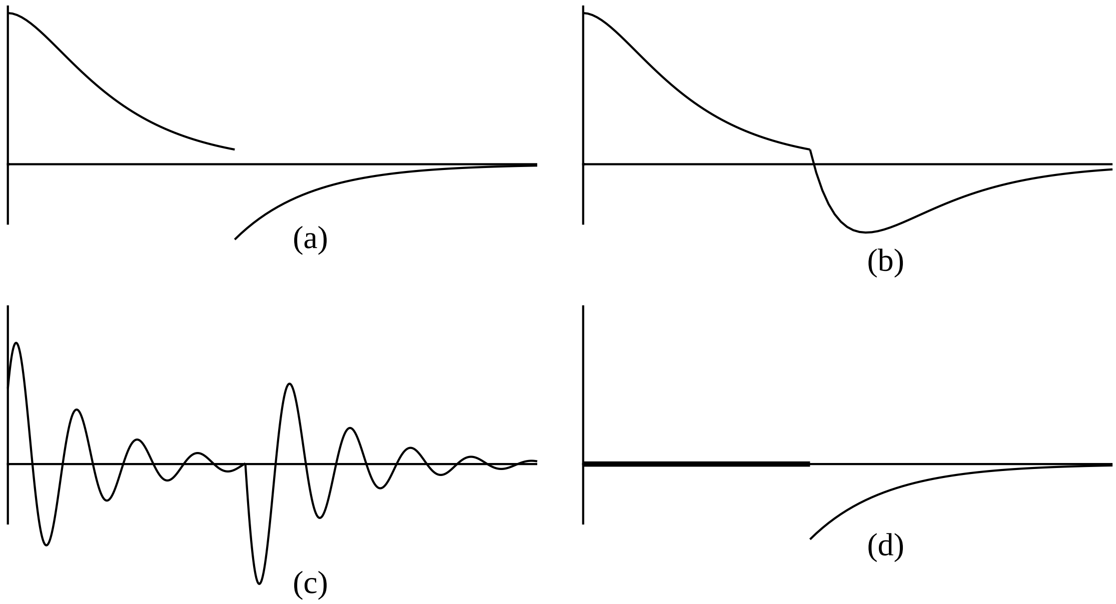
$$\boxed{y = 1 - 3e^{2t} + 2e^{4t}}$$

It's easy to check that this satisfies the initial conditions. It's also easy to see (using methods from Chapter 3) that this has the right form to be a solution.

3. Consider the initial value problem

$$y'' + 3y' + 2y = -4\delta(t-3),$$
$$y(0) = 2,\ y'(0) = 0.$$

Which of the following is the graph of the solution? (Hint: you can answer this question without solving the differential equation.)



(e) none of the above

Solution. The answer is (b). Since the impulse described by the delta function occurs at time $t = 3$, this acts just like the solution to

$$y'' + 3y' + 2y = 0, \quad y(0) = 2,\ y'(0) = 0$$

as long as $t < 3$. The solution to this equation is a sum of decaying exponentials, so you can eliminate (c) and (d). At $t = 3$, the system is given an impulse with magnitude 4 in the negative direction, and then it decays exponentially afterward. The solution must be continuous, so you can eliminate (a). The graph in (b) works, so you can eliminate (e).

4. Which of the following is $\mathcal{L}(f(t))$, where $f(t) = \begin{cases} t^2, & t < 2, \\ 4 & t \geq 2? \end{cases}$

(a) $\dfrac{2}{s^3} - e^{-2s}\dfrac{2}{s^3} + e^{-2s}\dfrac{4}{s^2}$ (b) $\dfrac{2}{s^3} - e^{-2s}\dfrac{4}{s}$ (c) $\dfrac{2}{s^3} - e^{-2s}\dfrac{2}{s^3} - e^{-2s}\dfrac{4}{s^2}$

(d) $\dfrac{2}{s^3} - e^{-2s}\dfrac{2}{s^3} + e^{-2s}\left(\dfrac{4}{s^2} - \dfrac{8}{s}\right)$ (e) none of the above

Solution. The answer is (c). You need to rewrite $f(t)$ in terms of the step function first:

$$f(t) = t^2 + u_2(t)(-t^2 + 4).$$

Next, you need to rewrite the last factor in terms of $t-2$. A little algebra yields

$$-t^2 + 4 = -(t-2)^2 - 4(t-2),$$

so

$$f(t) = t^2 + u_2(t)(-(t-2)^2 - 4(t-2)).$$

If you want to be really pedantic, you can rewrite this as

$$f(t) = t^2 + u_2(t)g(t-2),$$

where $g(t) = -t^2 - 4t$. Therefore the Laplace transform of $f(t)$ is

$$F(s) = \frac{2}{s^3} + e^{-2s}G(s) = \frac{2}{s^3} + e^{-2s}\left(-\frac{2}{s^3} - \frac{4}{s^2}\right).$$

5. (a) Use the definition of the Laplace transform to derive the formula

$$\mathcal{L}(f'(t)) = sF(s) - f(0).$$

(Assume that $\lim\limits_{t\to\infty} e^{-st} f(t) = 0$.)

Solution. Here are two approaches.

(1) By the definition of the Laplace transform,

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)dt.$$

Use integration by parts on this integral, with $u = e^{-st}$ and $dv = f'(t)dt$ (so $du = -se^{-st}dt$ and $v = f(t)$):

$$\begin{aligned}\int_0^\infty e^{-st} f'(t)dt &= e^{-st} f(t)|_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt \\ &= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\int_0^\infty e^{-st} f(t)dt \\ &= -f(0) + s\mathcal{L}(f(t)).\end{aligned}$$

(2) Let $F(s) = \mathcal{L}(f(t))$. By the definition of the Laplace transform,

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt.$$

Use integration by parts on this integral, with $u = f(t)$ and $dv = e^{-st}dt$ (so $du = f'(t)dt$ and $v = -\frac{1}{s}e^{-st}$):

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= -\frac{1}{s}e^{-st} f(t)|_0^\infty - \int_0^\infty \left(-\frac{1}{s}\right) e^{-st} f'(t)dt \\
&= -\lim_{t\to\infty} \frac{1}{s}e^{-st} f(t) + \frac{f(0)}{s} + \frac{1}{s}\int_0^\infty e^{-st} f'(t)dt \\
&= \frac{1}{s}\left(f(0) + \mathcal{L}(f'(t))\right).
\end{aligned}$$

Summarizing,

$$F(s) = \frac{1}{s}\left(f(0) + \mathcal{L}(f'(t))\right).$$

Now solve for $\mathcal{L}(f'(t))$:

$$\mathcal{L}(f'(t)) = sF(s) - f(0).$$

(b) (**extra credit**) Use the formula from part (a) to compute $u_c'(t)$, where c is a positive constant. (Hint: First use part (a) to compute the Laplace transform of $u_c'(t)$.)

Solution. By the formula in part (a),

$$\mathcal{L}(u_c'(t)) = s\mathcal{L}(u_c(t)) - u_c(0).$$

Since $c > 0$, $u_c(0) = 0$. From the table, $\mathcal{L}(u_c(t)) = e^{-cs}/s$. So

$$\mathcal{L}(u_c'(t)) = e^{-cs}.$$

To compute $u_c'(t)$, take inverse Laplace transforms. Since the inverse Laplace transform of e^{-cs} is $\delta(t-c)$,

$$\boxed{u_c'(t) = \delta(t-c)}.$$

(In fact, just using Math 124 stuff, $u_c(t)$ is discontinuous when $t = c$, and its derivative is zero unless $t = c$. When $t = c$, the derivative is undefined.)

Part II: earlier problems

6. **Do not solve the problems in parts (a)–(d).** Instead, identify the type of equation ("separable," "second order linear homogeneous with constant coefficients," things like that), and tell me what method (or methods) to use to solve it ("integrating factor," "undetermined coefficients," etc.). Give brief but complete answers.

(a) $(1+x)\dfrac{dy}{dx} + y = \cos x$.

Solution. First order linear. Solve using an integrating factor. (Note: this is not a separable equation.)

(b) $y'' + 2y' + 5y = e^{-t} \sec 2t$.

Solution. Second order linear nonhomogeneous with constant coefficients. Solve by using the characteristic equation to find y_h, and then use variation of parameters to find y_p.

(c) $y' = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right) y$ (where r, T, and K are constants)

Solution. Two correct answers: this is separable, so you can solve it by separating the variables and integrating. Also, it's autonomous, so instead of solving it, you can do qualitative analysis, find the critical points and evaluate each one for stability, etc.

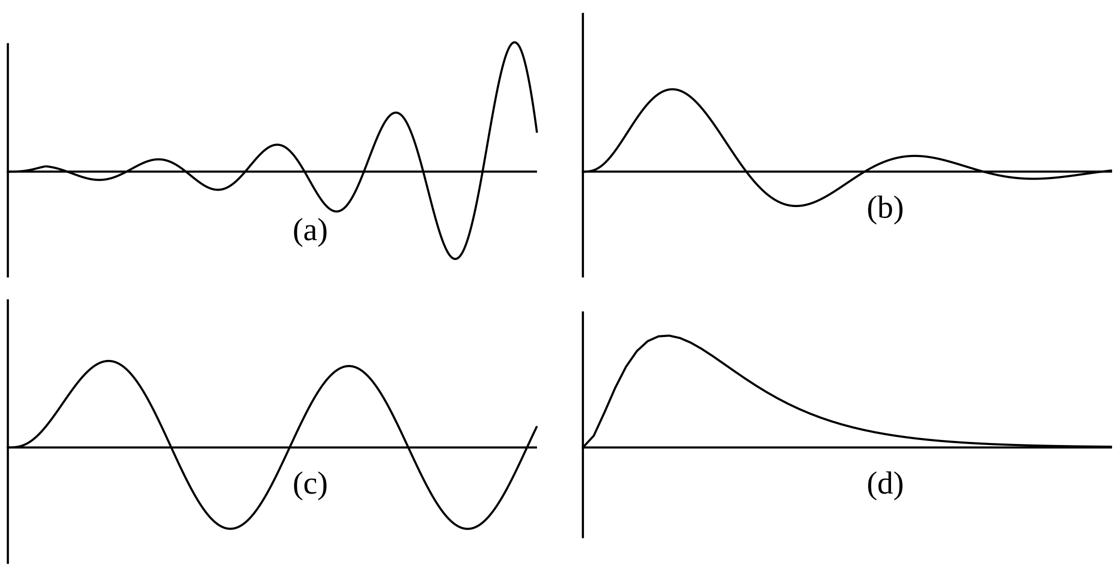
(d) $y'' + 6y' + 10y = 0$, $y(0) = 1$, $y'(0) = -1$.

Solution. Second order linear homogeneous with constant coefficients. Solve using the characteristic equation or Laplace transforms.

7. Consider the initial value problem

$$y'' + 6y' + 9y = \sin 2t,$$
$$y(0) = 0,\ y'(0) = 0.$$

Which of these is the graph of the solution? (Hint: you can answer this question without completely solving the differential equation.)



(e) none of the above

Solution. The answer is (c). Using the characteristic equation and undetermined coefficients, the answer will look like

$$y = c_1 e^{-3t} + c_2 t e^{-3t} + A\cos 2t + B \sin 2t,$$

for some numbers A and B (which can be found by the method of undetermined coefficients) and c_1 and c_2 (which can be found, once you know A and B, using the initial conditions). The exponential terms go to zero as t increases, so the solution should eventually look like the steady state solution $A\cos 2t + B\sin 2t$. Graph (c) is the best candidate – except for the part near the origin, this looks just like a sine or cosine curve.