Name: Solutions

Mathematics 307E&H Exam

Solutions

1. (a) (10 points) Find the general solution of $y'' + 3y' + 2y = 3e^t$.

Solution. First find y_h. The associated homogeneous equation is $y'' + 3y' + 2y = 0$; the characteristic equation is $r^2 + 3r + 2 = 0$, and this factors as $(r+1)(r+2) = 0$. So the roots are -1 and -2, and $y_h = c_1 e^{-t} + c_2 e^{-2t}$.

Next find y_p. Try $y_p = Ae^t$. This is not part of y_h, so it will work. $y_p' = Ae^t = y_p''$, so it's easy to plug in: $Ae^t + 3Ae^t + 2Ae^t = 3e^t$, so $6Ae^t = 3e^t$, so $6A = 3$, so $A = 1/2$. Thus $y_p = \frac{1}{2}e^t$, and the general solution is

$$\boxed{y = c_1 e^{-t} + c_2 e^{2t} + \frac{1}{2}e^t}.$$

(b) (10 points) Find the general solution of $y'' + 3y' = 0$.

Solution. Use the characteristic equation: for this equation, it's $r^2 + 3r = 0$, which factors as $r(r+3) = 0$, so it has roots 0 and -3. So the general solution is

$$y = c_1 e^{0t} + c_2 e^{-3t},$$

which simplifies to

$$\boxed{y = c_1 + c_2 e^{-3t}}.$$

(c) (10 points) Find the solution of $y'' + 3y' - 3y = 0$.

Solution. The characteristic equation is $r^2 + 3r - 3 = 0$. Use the quadratic formula to find the roots:

$$r = \frac{-3 \pm \sqrt{9 - 4 \cdot (-3)}}{2} = \frac{-3 \pm \sqrt{21}}{2}.$$

So there are two real roots, and the general solution is

$$\boxed{y = c_1 e^{\frac{-3+\sqrt{21}}{2}t} + c_2 e^{\frac{-3-\sqrt{21}}{2}t}}.$$

2. Here is a nonhomogeneous differential equation:

$$y'' + 2y' + y = g(t).$$

(a) (5 points) What is y_h, the solution of the associated homogeneous equation?

Solution. The characteristic equation is $r^2+2r+1=0$, which is the same as $(r+1)^2=0$. It has only one root, $r=-1$, so $\boxed{y_h = c_1e^{-t}+c_2te^{-t}}$.

For the remaining parts, I'll tell you $g(t)$, and I want you to tell me what to try for y_p, according to the method of undetermined coefficients. You don't have to solve for the coefficients, just tell me the right form. For full credit, take your answer for part (a) into account.

(b) (5 points) If $g(t)=3\sin 2t$, what should you try for y_p?

Solution. Try $\boxed{y_p = A\sin 2t + B\cos 2t}$. This is not part of y_h, so it will work.

(c) (5 points) If $g(t)=6e^t$, what should you try for y_p?

Solution. Try $\boxed{y_p = Ae^t}$. This is not part of y_h, so it will work.

(d) (5 points) If $g(t)=-3e^{-t}\cos 2t + t^2$, what should you try for y_p?

Solution. Try $\boxed{y_p = Ae^{-t}\cos 2t + Be^{-t}\sin 2t + Ct^2 + Dt + E}$. This is not part of y_h, so it will work. (If it had just had Ae^{-t}, it would have been part of y_h. $e^{-t}\cos 2t$ is not part of y_h, though.)

3. (a) (10 points) Here is the equation of a damped mass-spring system with an external force acting on it: $u''+0.1u'+9u=5\cos\omega t$. For what value(s) of ω, approximately, will the system exhibit its largest oscillations?

Solution. The system has its largest oscillations when the driving frequency ω is close to, but a bit smaller than, the natural frequency ω_0. The natural frequency is $\omega_0 = \sqrt{k/m} = \sqrt{9/1} = 3$ in this case, so $\boxed{\omega \text{ should be just a bit smaller than } 3}$ to get the largest possible oscillations. (To get the precise answer, you take the formula for the amplitude of the oscillations and maximize it. I only asked for approximate values, so this was not necessary.)

(b) (10 points) Here is the equation of a damped mass-spring system with no external force acting on it: $u''+\gamma u'+5u=0$. For what value(s) of γ will the mass oscillate?

Solution. The mass will oscillate when the solution has sines and cosines in it, which happens when the characteristic equation has complex roots. (Equivalently, this happens when the system is underdamped.) This happens when the piece of the quadratic formula under the square root (the "discriminant") is negative. In this case, the discriminant is $\gamma^2-4\cdot 5=\gamma^2-20$. This will be negative when $\gamma^2<20$, which happens when $\boxed{-\sqrt{20}<\gamma<\sqrt{20}}$. (In spring problems, γ can never be negative, so it is also correct to say $\boxed{0\le\gamma<\sqrt{20}}$.)

4. (15 points) Find the general solution of $t^2y''-t(t+2)y'+(t+2)y=2t^3$, by taking the following steps.

(a) Verify that $y_1 = t$ is a solution to the associated homogeneous equation.

Solution. If $y_1 = t$, then $y_1' = 1$ and $y_1'' = 0$. Plug these in to the associated homogeneous equation: the left-hand side is $t^2 0 - t(t+2)1 + (t+2)t$, and this is indeed zero, the way it should be.

(b) Find y_h, the general solution to the associated homogeneous equation.

Solution. Since the equation does not have constant coefficients, I can't use the characteristic equation. On the other hand, I know one solution, $y_1 = t$, so I can use reduction of order to find another one. So let $y_2 = vy_1 = vt$; then $y_2' = v't + v$ and $y_2'' = v''t + v' + v' = v''t + 2v'$. Plug this into the associated homogeneous equation:

$$\begin{gathered} t^2(tv'' + 2v') - t(t+2)(tv' + v) + (t+2)(tv) = 0 \\ t^3v'' + 2t^2v' + (-t^2 - 2t)(tv' + v) + t^2v + 2tv = 0 \\ t^3v'' + 2t^2v' - t^3v' - 2t^2v' - t^2v - 2v + t^2v + 2tv = 0 \end{gathered}$$

Lots of thing cancel here, leaving

$$\begin{gathered} t^3v'' - t^3v' = 0 \\ v'' - v' = 0. \end{gathered}$$

Let $w = v'$, so $w' = v''$. Then this becomes $w' - w = 0$, or $w' = w$. This is easy to solve; the solution is $v' = w = Ae^t$. So $v = Ae^t + c$, and $y_2 = tv = Ate^t + ct$. The general solution is then

$$y_h = c_1t + c_2(Ate^t + ct).$$

I can rearrange and rename constants to write this as

$$\boxed{y_h = c_1t + c_2te^t}.$$

(Alternatively, I can let $A = 1$ and $c = 0$.) In particular, I know that $y_1 = t$, and I can let $y_2 = te^t$.

(c) Find y_p, a particular solution to the nonhomogeneous equation. (Hint: since this equation doesn't have constant coefficients, don't use the method of undetermined coefficients.)

Solution. My only option is variation of parameters. With $y_1 = t$ and $y_2 = te^t$,

$$W = t(te^t)' - (t)'te^t = t(te^t + e^t) - te^t = t^2e^t.$$

To use the variation of parameters formula, the coefficient of y'' must be one, so divide the equation by t^2. The new equation is

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 2t.$$

So y_p is given by this equation:

$$\begin{aligned}
y_p &= -t\int \frac{(te^t)(2t)}{t^2e^t}dt + te^t\int \frac{(t)(2t)}{t^2e^t}dt \\
&= -t\int 2\,dt + te^t\int \frac{2}{e^t}dt \\
&= -t2t + 2te^t\int e^{-t}dt \\
&= -2t^2 + 2te^t(-e^{-t}) \\
&= -2t^2 + 2t.
\end{aligned}$$

(d) Assemble to get the general solution to the nonhomogeneous equation. Simplify your answer for full credit.

Solution. The general solution is $y = y_h + y_p$, which is

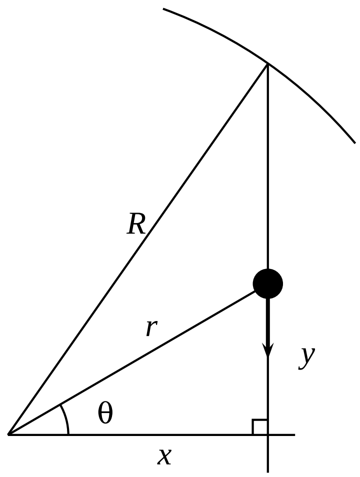
$$y = c_1t + c_2te^t - 2t^2 + 2t.$$

I can absorb the $2t$ term into the c_1t term, so the simplest form is

$$\boxed{y = c_1t + c_2te^t - 2t^2}.$$

5. (15 points) Imagine a train which goes in a straight line through a tunnel between two points on the surface of the earth. Suppose that there is no friction, and the train just operates under gravity, so if you start it at one end of the tunnel, at rest, it will fall into the tunnel, accelerate until it reaches the midpoint of its trip, decelerate as it nears the other end, stop at the other end, and then return. What is the fastest that such a train will ever go? Express your answer in terms of familiar constants (e.g., gravitational acceleration g, the radius of the earth R, the mass of the electron, Ichiro's batting average, etc.).

[Hints: Inside the earth, the force of gravity is proportional to the distance from the center. Let y be the distance from the train to the midpoint of the tunnel, set up a differential equation for y, and solve it. Then compute the velocity y' from that.]



Solution. Note that $y = r \sin\theta$.

The force on the train is due to gravitation, and according to the hint, its magnitude is $-mkr$, where k is some constant. Actually, I can work out what k is, because I know that at the surface of the earth – that is, when $r = R$ – the gravitational force is $-mg$. Therefore $-mkR = -mg$, so $k = g/R$. The direction of this force is in a line from the train to the center of the earth. Now, the train is being forced to travel along its tunnel, so I really only care about the component of the force in that direction. Looking at the triangle in the picture, the gravitational force is acting along the radial line of length r; the component in the "y-direction" is just $\sin\theta$ multiplied by the force. So the force acting on the train is actually:

$$-\frac{mgr}{R}\sin\theta.$$

So Newton's second law says

$$my'' = -\frac{mgr}{R}\sin\theta = -\frac{mg}{R} r \sin\theta.$$

But $y = r\sin\theta$, so this says

$$my'' = -\frac{mg}{R} y.$$

In other words, I have this differential equation:

$$my'' + \frac{mg}{R} y = 0,$$

or equivalently,

$$y'' + \frac{g}{R} y = 0.$$

This is a second order linear homogeneous differential equation with constant coefficients (since g and R are both constants), and so it's easy to solve. The characteristic equation is

$$s^2 + \frac{g}{R} s = 0$$

(I'm using s because I've already used r in this problem for something else). The roots are $s = \pm i\sqrt{g/R}$, and so the general solution is

$$y = c_1 \cos\left(\sqrt{\frac{g}{R}}\,t\right) + c_2 \sin\left(\sqrt{\frac{g}{R}}\,t\right).$$

[Up to this point, this is identical to the homework problem.]

The speed of the train is

$$y' = -c_1\sqrt{\frac{g}{R}} \sin\left(\sqrt{\frac{g}{R}}\,t\right) + c_2\sqrt{\frac{g}{R}} \cos\left(\sqrt{\frac{g}{R}}\,t\right).$$

In order to know when this is largest, I'd better find c_1 and c_2. The train starts at rest, so $y'(0) = 0$. It also starts at one end of the tunnel, so by the Pythagorean theorem, $y(0) = \sqrt{R^2 - x^2}$. (Remember, x is constant for any particular tunnel. r, y, and θ all change as the train moves.) So $c_1 = \sqrt{R^2 - x^2}$ and $c_2 = 0$, and the position and velocity of the train are given by the formulas

$$y = \sqrt{R^2 - x^2} \cos\left(\sqrt{\frac{g}{R}}\,t\right),$$
$$y' = -\sqrt{R^2 - x^2}\sqrt{\frac{g}{R}} \sin\left(\sqrt{\frac{g}{R}}\,t\right).$$

I want to know the largest possible velocity. Since sine oscillates between plus and minus one, this is largest when the sine part equals -1. So the largest velocity is

$$\sqrt{R^2 - x^2}\sqrt{\frac{g}{R}}.$$

This is not quite the full story, though. If I look at all possible tunnels, which is to say, all possible values of x, this is largest when x is zero (which makes sense – the train straight through the center of the earth will be the fastest one). So the largest possible velocity is

$$\sqrt{R^2}\sqrt{\frac{g}{R}} = \boxed{\sqrt{gR}}.$$

(If you actually plug in numbers for this, you get $7{,}910$ meters per second, which is the same as $28{,}500$ kph or $17{,}700$ mph.)