Name: Solutions

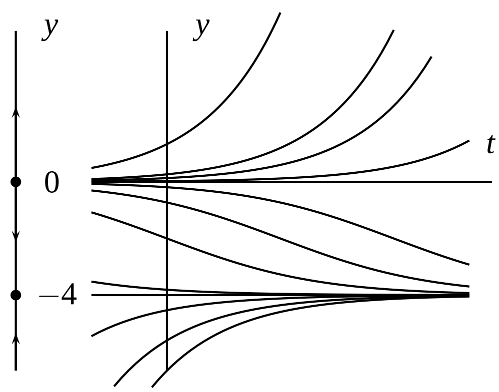
Mathematics 307E&H Exam

Solutions

1. Consider the differential equation $y' = 2y(y+4)(y^2+1)$.

   (a) (15 points) Determine the equilibrium points, classify each one as asymptotically stable, semistable, or unstable, and sketch some of the solution curves.

   Solution. y' is zero when $y = 0$, $y+4 = 0$, or $y^2+1 = 0$. This last term is never zero, so the equilibrium points are $\boxed{y = 0}$ and $\boxed{y = -4}$. When $y < -4$, y' is positive; when $-4 < y < 0$, y' is negative; and when $y > 0$, y' is positive. Therefore $y = -4$ is $\boxed{\text{asymptotically stable}}$ and $y = 0$ is $\boxed{\text{unstable}}$. Here's a graph:



   (b) (5 points) If $y(0) = 2$, compute $\lim_{t\to\infty} y$ and $\lim_{t\to-\infty} y$.

   Solution. From the graph in part (a), if $y(0) = 2$, then maybe that's the top curve in the graph, so y approaches infinity as t moves to the right, and it approaches zero as t goes to the left:

$$\boxed{\lim_{t\to\infty} y = \infty, \quad \lim_{t\to-\infty} y = 0}.$$

2. Solve these differential equations and initial value problems. Be sure to check your work.

   (a) (15 points) $y' + y = 2t$, $y(0) = 1$.

   Solution. This is a first order linear equation. To find the integrating factor, I integrate the coefficient of y (which is 1, so its integral is t) and then exponentiate: e^t is the integrating factor. Multiply by it:

$$e^t y' + e^t y = 2te^t.$$

   Now integrate: the integral of the left side is the integrating factor times y:

$$e^t y = 2 \int te^t \, dt.$$

   Do the remaining integral by parts, or use the formula on the last page of the exam; by either method, you should get $\int te^t \, dt = te^t - e^t$. So the equation becomes

$$\begin{aligned} e^t y &= 2(te^t - e^t + c) \\ e^t y &= 2te^t - 2e^t + c \\ y &= 2t - 2 + ce^{-t}. \end{aligned}$$

   When $t = 0, y = 1$. When I plug these values in, I get $1 = -2 + c$, so $c = 3$, and the answer is

$$\boxed{y = 2t - 2 + 3e^{-t}}.$$

   (b) (15 points) $\dfrac{dy}{dx} = \dfrac{e^x + 2x}{y}$.

   Solution. This is a separable equation, so separate the variables and integrate:

$$\begin{aligned} \int y \, dy &= \int (e^x + 2x) dx \\ \frac{1}{2} y^2 &= e^x + x^2 + c \\ y^2 &= 2e^x + 2x^2 + c \\ &\boxed{y = \pm\sqrt{2e^x + 2x^2 + c}} \end{aligned}$$

3. (a) (15 points) Suppose you have a large tank which initially contains 50 liters of a solution, and the solution consists of water with 10 grams of salt dissolved in it (so its initial concentration is $10/50$ g/l). Starting at time $t = 0$, salt water with concentration 5 g/l is pumped in, at a rate of 2 liters per minute. The well-mixed solution is pumped out of the tank at a rate of 1 liter per minute. Find a formula for the amount of salt in the tank at any time t.

Solution. First I'll deal with the volume in the tank. Since it starts at 50, and since I pump in 2 liters per minute and pump out 1 liter per minute, the volume at time t is $V = 50+t$. Now let y be the amount of salt in the tank at time t. The rate at which salt is being added is 2 liters per minute times 5 grams per liter; the rate at which salt is being removed is 1 liter per minute times the concentration in the tank. Thus The relevant equation for y is

$$\begin{aligned}\frac{dy}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= 2 \cdot 5 - 1 \cdot \frac{y}{V} \\ &= 10 - \frac{y}{t+50}.\end{aligned}$$

So this is the differential equation I have to solve: $y' = 10 - y/(t+50)$. Rearranging things a bit, I have

$$y' + \frac{1}{t+50} y = 10.$$

This is first order linear; to find the integrating factor, identify the coefficient of y ($1/(t+50)$), integrate it ($\ln(t+50)$), and exponentiate the result (giving $t+50$). Multiply the equation by that:

$$(t+50)y' + y = 10t + 500.$$

Integrate and solve for y:

$$(t+50)y = 5t^2 + 500t + cy = \frac{5t^2 + 500t + c}{t+50}$$

The initial condition $y(0) = 10$ yields $c = 500$, so the answer is

$$\boxed{y = \frac{5t^2 + 500t + 500}{t+50}}.$$

(b) (10 points) A mothball evaporates (loses volume) at a rate proportional to its surface area. If its initial radius is 1.5 centimeter, and after 1 month its radius is 1 centimeter, what is its radius after 2 months? (Hint: the chain rule says that $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$; use this to find a differential equation for the radius r.)

Solution. Since the surface area of a sphere of radius r is $4\pi r^2$, the first sentence says that $\frac{dV}{dt} = -k4\pi r^2$. Since the volume of a sphere is $V = \frac{4}{3}\pi r^2$, I can compute $\frac{dV}{dr}$ by differentiation: $\frac{dV}{dr} = 4\pi r^2$. The chain rule says that $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$, so I have

$$\frac{dV}{dt} = -k4\pi r^2 \quad \text{and} \quad \frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Therefore

$$-k4\pi r^2 = 4\pi r^2 \frac{dr}{dt},$$

so $\frac{dr}{dt} = -k$. This is an easy equation to solve: $r = -kt + c$. From the initial condition $r(0) = 1.5$, I get $c = 1.5$. From the condition $r(1) = 1$, I get $k = 0.5$. Thus $r = -0.5t + 1.5$. Plug in $t = 2$ to get the answer: after two months, $\boxed{r = 0.5}$.

4. (10 points) Consider the initial value problem $y' = \ln(y+t)$, $y(1) = 0$. Use Euler's method, with step size $h = 1$, to approximate $y(2)$ and $y(3)$.

   Solution. I am told that when $t = 1$, $y = 0$. Let's make these the starting values for Euler's method: $t_0 = 1$, $y_0 = 0$. Using step size $h = 1$, my next t-value is $t_1 = 1 + 1 = 2$. My next y-value is

   $$y_1 = y_0 + hf(y_0, t_0) = 0 + 1 \cdot \ln(0+1) = \ln 1 = 0.$$

   Therefore $\boxed{y(2) \approx 0}$.

   My next t-value is $t_2 = 2 + 1 = 3$. My next y-value is

   $$y_2 = y_1 + hf(y_1, t_1) = 0 + 1 \cdot \ln(0+2) = \ln 2.$$

   Therefore $\boxed{y(3) \approx \ln 2}$.

5. (a) (10 points) State Euler's formula (the one about complex numbers).

   Solution. $\boxed{e^{it} = \cos t + i \sin t}$

   (b) (5 points) Use Euler's formula to write $e^{\ln 2 + i\pi/2}$ in the form $a + bi$. For full credit, simplify as much as possible.

   Solution.

   $$\begin{aligned} e^{\ln 2 + i\pi/2} &= e^{\ln 2} e^{i\pi/2} \\ &= 2\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) \\ &= 2(0 + i \cdot 1) \\ &= \boxed{2i}. \end{aligned}$$

   (If you really want to write it in the form $a + bi$, you could say $0 + 2i$.)