Homology and cohomology

April 22, 2002

Definition 0.1. A *pair* of CW complexes is a pair (X, A) of spaces with $A \subseteq X$.

Definition 0.2 (Eilenberg-Steenrod). A *homology theory* h assigns to each pair (X, A) a sequence of abelian groups $h_n(X, A)$, $n \in \mathbf{Z}$, satisfying the following conditions:

- (Functoriality.) Each h_n is a functor. That is, given a map $f : X \to Y$ carrying the subspace A to B, there are induced homomorphisms $h_n f : h_n(X, A) \to h_n(Y, B)$, satisfying the appropriate conditions (composition, identity).
- (Homotopy axiom.) If $f \simeq g : X \to Y$, then $h_n f = h_n g$ for all n.
- (Exact sequences.) Write $h_n X$ for $h_n(X, \emptyset)$. For any pair (X, A), there are natural "boundary homomorphisms" $\partial : h_n(X, A) \to h_{n-1} A$ so that the sequence

  $$\cdots \to h_n A \xrightarrow{h_n i} h_n X \xrightarrow{h_n j} h_n(X, A) \xrightarrow{\partial} h_{n-1} A \to \cdots$$

  is exact. In this diagram, i is the inclusion map $i : A \hookrightarrow X$, and j is the map of pairs $(X, \emptyset) \to (X, A)$ induced by the identity map on X.
- (Excision.) For any CW pair (X, A), $h_n(X, A) \cong h_n(X/A, A/A)$.
- (Sums.) Given spaces X_α with subspaces $A_\alpha \subseteq X_\alpha$, let $X = \coprod X_\alpha$ and $A = \coprod A_\alpha$. Then the map

  $$\bigoplus h_n i_\alpha : \bigoplus h_n(X_\alpha, A_\alpha) \to h_n(X, A)$$

  is an isomorphism, where $i_\alpha X_\alpha \hookrightarrow X$ is the inclusion map.

For most of the cases we're going to be concerned with, $h_n(\text{point})$ will be nonzero only when $n = 0$; in this case, the group $h_0(\text{point}) = G$ is called the *coefficients* of the homology theory. The resulting theory is called *homology with coefficients in* G, is written $H_n(X, A; G)$, and is uniquely determined by the definition and G.

Definition 0.3. A *cohomology theory* is "dual" to this. It assigns to each pair (X, A) a sequence of abelian groups $h^n(X, A)$, $n \in \mathbf{Z}$, satisfying the following conditions:

- (Functoriality.) Each h^n is a contravariant functor.
- (Homotopy axiom.) If $f \simeq g : X \to Y$, then $h^n f = h^n g$ for all n.
- (Exact sequences.) For any pair (X, A), there are natural "boundary homomorphisms" $\delta : h^n A \to h^{n+1}(X, A)$ so that the sequence

  $$\cdots \to h^n(X, A) \xrightarrow{h^n j} h^n X \xrightarrow{h^n i} h^n A \xrightarrow{\delta} h^{n+1}(X, A) \to \cdots$$

  is exact.
- (Excision.) For any CW pair (X, A), $h^n(X, A) \cong h^n(X/A, A/A)$.
- (Products.) Given spaces X_α with subspaces $A_\alpha \subseteq X_\alpha$, let $X = \coprod X_\alpha$ and $A = \coprod A_\alpha$. Then the map

  $$\prod h^n i_\alpha : h^n(X, A) \to \prod h^n(X_\alpha, A_\alpha)$$

  is an isomorphism.

Coefficients work the same way as for homology.

An important feature of cohomology: if R is a commutative ring, then cohomology with coefficients in R, $H^*(-; R) = \bigoplus H^i(-; R)$, has the structure of a graded commutative R-algebra. For any space X, there is a unique map $X \to (\text{point})$; in cohomology, this induces $H^*(\text{point}; R) \to H^*(X; R)$, and the image of $1 \in R = H^0(\text{point}; R)$ is the unit element in $H^*(X; R)$. One can show that if X is connected, then this map in cohomology is an isomorphism in $H^0(-; R)$ (and in fact in $H^i(-; R)$ for all $i \leq 0$).

By the way, the product is called the "cup product".

Example 0.4. Let R be a commutative ring.

- First, if X is the one point space, then $H^i(X; R)$ is given by

  $$H^i(X; R) = \begin{cases} R, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

  This is by definition of the cohomology theory $H^*(-; R)$.
- Next, the cohomology algebras of spheres. I claim that $H^i(S^n; R)$ is nonzero only when $i = 0$ and $i = n$: if $n = 0$, then $H^0(S^0; R) = R \oplus R$, and if $n > 0$, then $H^0(S^n; R) = H^n(S^n; R) = R$. The proof is by induction on n, using excision and the long exact sequence (and the product axiom for the $n = 0$ case).

The algebra structure is essentially determined by this. When $n > 0$,

$$H^i(X; R) = \begin{cases} R & \text{if } i = 0 \text{ (unit element)}, \\ R & \text{if } i = n \text{ (spanned by some element } x), \\ 0 & \text{otherwise.} \end{cases}$$

Because of the grading, x^2 is in $H^{2n}(S^n; R) = 0$. Thus all products are zero here: as a graded algebra $H^*(S^n; R) = R[x]/(x^2)$, with x in degree n.

I'll leave the ring structure when $n = 0$ to the reader.

- Other examples: $H^*(\mathbf{C}P^n; R) = R[x]/(x^{n+1})$, where x is in degree 2. If $R = \mathbf{Z}$ or if R is a field, and if X and Y are spaces with, say, $H^i(X; R)$ free for each i, then $H^*(X \times Y; R) \cong H^*(X; R) \otimes_R H^*(Y; R)$. So for example, $H^*(S^1 \times S^1; \mathbf{Z}) = \Lambda(x, y)$, where x and y are in degree 1. Here, $\Lambda(x, y) = \Lambda_{\mathbf{Z}}(x, y)$ denotes the exterior algebra on x and y: the free $\mathbf{Z}$-algebra on those generators, subject to the relations $ab = (-1)^{(\deg a)(\deg b)} ba$ for all a and b.

- If $R = \mathbf{F}_2$, then $H^*(\mathbf{R}P^n; \mathbf{F}_2) \cong \mathbf{F}_2[x]/(x^{n+1})$.