Fibrations
from McCleary, Section 4.3

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The important thing about fibrations, as far as this course is concerned, is the examples. I'll give definitions and results, too.

Definition 0.1. A map $p : E \to B$ has the *homotopy lifting property* with respect to a space Y if, given a map $G : Y \times I \to B$ and a map $g : Y \to E$ so that $p \circ g$ is the restriction of G to the 0-end of $Y \times I$, then there is a map $\widetilde{G} : Y \times I \to E$ lifting G:

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{g} & E \\ \big\downarrow & \overset{\widetilde{G}}{\nearrow} & \big\downarrow p \\ Y \times I & \xrightarrow{G} & B \end{array}$$

A *(Hurewicz) fibration* is a map $p : E \to B$ which has the homotopy lifting property with respect to all spaces. A *Serre fibration* is a map which has the homotopy lifting property (HLP) with respect the n-disk for all $n \geq 0$. (Equivalently, a Serre fibration has the HLP with respect to all finite CW complexes.)

For either kind of fibration, E is called the *total space* and B the *base space*.

Given a point $b \in B$, then $F_b = p^{-1}(b)$ is the *fiber* of p at b.

Proposition 0.2. *Given a fibration $p : E \to B$, if b_0 and b_1 are points in B which are connected by a path, then F_{b_0} is homotopy equivalent to F_{b_1}.*

(So any reasonable algebraic invariant gives the same answer for F_{b_0} and F_{b_1}.)

Because of this, we will often want to assume that the base space is path-connected. Also because of this, we will often refer to "the fiber" of a fibration − this is not well-defined up to homeomorphism, but it is up to homotopy equivalence.

Definition 0.3. Given any space B with basepoint b, let PB denote the space of paths in B starting at b: $PB = \{\gamma : I \to B \,|\, \gamma(0) = b\}$. Then PB can be given a reasonable topology (the "compact-open" topology), and if B is path-connected, PB is contractible (homotopy equivalent to a point). Let ΩB denote the *(based) loop space* of B: the subspace of PB consisting of all paths starting and ending at b.

(One can also define the spaces of unbased paths and loops, but I'm not going to discuss those.)

Example 0.4. 1. Any fiber bundle $E \to B$ with B paracompact is a Hurewicz fibration. Any fiber bundle is a Serre fibration. In this case, of course, the fibers over any two points are homeomorphic (if B is connected).

2. The projection map $F \times B \to B$ is a fibration, called the "trivial fibration". The fiber over every point is homeomorphic to F.

3. Covering spaces are fibrations. The fiber over every point is discrete.

4. Given a space X with basepoint x_0, the map $PX \to X$ defined by $\gamma \mapsto \gamma(1)$ is a fibration. The fiber over x_0 is ΩX; the fiber over any other point x_1 is the space of paths from x_0 to x_1, which is homotopy equivalent to ΩX as long as X is path-connected. This is called the "path-loop fibration".

5. If G is a Lie group and $H \leq G$ a closed subgroup, then $G \to G/H$ is a fibration with fiber H.

6. The Hopf fibrations: for $n = 1, 2, 4, 8$, there are fibrations $S^{2n-1} \to S^n$ with fiber S^{n-1}. When $n = 1$, for instance, this is a map from the unit sphere in $\mathbf{R}^2$ to one-dimensional real projective space; when $n = 2$, it's the analogous thing with the complex numbers, etc.

One more topic: given a fibration $E \to B$ with fiber F, there is an action of $\pi_1(B)$ on the homology and cohomology of F, and I want to define that.

ΩB is an "H-space": that is, it's a topological group up to homotopy. What I mean is this: there is a product $\Omega B \times \Omega B \to \Omega B$, defined by concatenating loops, which is associative and commutative up to homotopy, and which has an identity element and inverses up to homotopy. It's a good idea to think of ΩB as being a topological group; indeed, it is homotopy equivalent to one.

Fix a fibration $p : E \to B$ with fiber $F = F_b$. There is an action of ΩB on $F = F_b$; given a point $y \in F \subseteq E$ and a loop γ based at b, one can use the definition of fibration to lift γ to a path $\widetilde{\gamma}$ in E starting at y and ending at some point, say $z \in E$. Since $\widetilde{\gamma}$ is a lift of γ, the end point z must map to b, and hence is in the fiber F_b. So the action is

$$\begin{aligned} \mu : \Omega B \times F &\longrightarrow F, \\ (\gamma, y) &\longmapsto z. \end{aligned}$$

For any $\gamma \in \Omega B$, let $h_\gamma(w) = \mu(\gamma^{-1}, w)$. Then h_γ is a map from F to itself.

Proposition 0.5. *1. If $\alpha \simeq \beta$, then $h_\alpha \simeq h_\beta$.*

2. If α is the constant loop at b, then $h_\alpha = 1_F$.

3. If $\alpha * \beta$ is the loop multiplication of α and β, then $h_{\alpha*\beta} = h_\alpha \circ h_\beta$.

Corollary 0.6. *Let G be an abelian group and $p : E \to B$ a fibration. Assume that B is path-connected and fix $b \in B$. Then h_- induces an action of $\pi_1(B, b)$ on $H_*(F; G)$ and on $H^*(F; G)$.*