An approach to the Bockstein spectral sequence

Notation

Let $C^\bullet$ be a chain complex of $\mathbf{Z}$-modules, and for any abelian group G, let $C^\bullet(G) = C^\bullet \otimes G$. Write $H^*(G)$ for the homology of $C^\bullet(G)$.

Computing homology with coefficients in $\mathbf{Q}/\mathbf{Z}_{(p)}$

The sequence of inclusions

$$\mathbf{Z}/p \hookrightarrow \mathbf{Z}/p^2 \hookrightarrow \cdots \hookrightarrow \mathbf{Q}/\mathbf{Z}_{(p)}$$

induces a filtration

$$C^\bullet(\mathbf{Z}/p) \hookrightarrow C^\bullet(\mathbf{Z}/p^2) \hookrightarrow \cdots \hookrightarrow C^\bullet(\mathbf{Q}/\mathbf{Z}_{(p)})$$

so that $C^\bullet(\mathbf{Q}/\mathbf{Z}_{(p)}) = \varinjlim C^\bullet(\mathbf{Z}/p^n)$. Since homology commutes with direct limits, $H^*(\mathbf{Q}/\mathbf{Z}_{(p)}) = \varinjlim H^*(\mathbf{Z}/p^n)$.

More precisely, define a decreasing filtration F^* on $C^\bullet(\mathbf{Q}/\mathbf{Z}_{(p)})$ by $F^sC^\bullet = C^\bullet(\mathbf{Z}/p^{-s})$ when $s \leq 0$, and $F^sC^\bullet = 0$ when $s \geq 0$. Then for each $s < 0$, there is a short exact sequence

$$0 \to F^{s+1}C^\bullet \to F^sC^\bullet \to F^sC^\bullet/F^{s+1}C^\bullet \to 0,$$

and furthermore, $F^sC^\bullet/F^{s+1}C^\bullet \cong C^\bullet(\mathbf{Z}/p)$. Taking homology gives a spectral sequence with

$$E_1^{s,t} = H^{s+t}(F^sC^\bullet/F^{s+1}C^\bullet) \cong \begin{cases} H^{s+t}(\mathbf{Z}/p), & \text{when } s < 0, \\ 0 & \text{when } s \geq 0, \end{cases}$$

converging to $H^{s+t}(\mathbf{Q}/\mathbf{Z}_{(p)})$, with $d^r : E_r^{s,t} \to E_r^{s+r,t-r+1}$. d^r is the rth Bockstein operation.

Note that $E_1^{s,t} = 0$ unless $s < 0$ and (if $C^\bullet$ is zero in negative dimensions) $s + t \geq 0$, so it's a "third octant" spectral sequence – it's a second quadrant spectral sequence, only nonzero above the line $s + t = 0$. The differentials are periodic, and are determined by their image in $E_r^{0,t}$. Also, the extension problems are solved: given an $x \in H^n(\mathbf{Z}/p)$, if x supports no differentials and is not in the image of any differential, then it produces a sequence of summands $\mathbf{Z}/p \subseteq E_\infty^{-1,n+1}$, $\mathbf{Z}/p \subseteq E_\infty^{-2,n+2}$, $\mathbf{Z}/p \subseteq E_\infty^{-3,n+3}$, ..., which fit together to form a copy of $\mathbf{Q}/\mathbf{Z}_{(p)}$. If x supports a d_r, then there is a corresponding $\mathbf{Z}/p$ summand in $E_\infty^{-i,n+i}$ for $1 \leq i \leq r$, and these fit together to form a copy of $\mathbf{Z}/p^r$ in the target.

Finally, by an application of the universal coefficient theorem, or by looking at the short exact sequences

$$0 \to \mathbf{Z} \to \mathbf{Z}_{(p)} \to \mathbf{Z}_{(p)}/\mathbf{Z} \to 0,$$
$$0 \to \mathbf{Z}_{(p)} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}_{(p)} \to 0,$$

one can relate $H^*(\mathbf{Q}/\mathbf{Z}_{(p)})$ to $H^*(\mathbf{Z})/(\text{torsion prime to } p)$: knowing either one lets you compute the other.

Computing homology with coefficients in $\mathbf{Z}_p^\wedge$

The sequence of surjections

$$\mathbf{Z}_p^\wedge \twoheadrightarrow \cdots \twoheadrightarrow \mathbf{Z}/p^3 \twoheadrightarrow \mathbf{Z}/p^2 \twoheadrightarrow \mathbf{Z}/p$$

induces a filtration

$$C^\bullet(\mathbf{Z}_p^\wedge) \to \cdots \to C^\bullet(\mathbf{Z}/p^3) \to C^\bullet(\mathbf{Z}/p^2) \to C^\bullet(\mathbf{Z}/p),$$

but $C^\bullet(\mathbf{Z}_p^\wedge) \neq \varprojlim C^\bullet(\mathbf{Z}/p^n)$ in general. These will be equal if, for example, $C^\bullet$ is the cellular chain complex for a locally finite cell complex.

One can use this filtration, as above, to set up a spectral sequence with

$$E_1^{s,t} = H^{s+t}(\mathbf{Z}/p) \Rightarrow H^{s+t}(\mathbf{Z}_p^\wedge),$$
$$d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}.$$

Note that $E^1_{s,t} = 0$ unless $s \geq 0$ and (if $C^\bullet = 0$ in negative dimensions) $s+t \geq 0$ – it's a right half-plane spectral sequence, concentrated above the line $s+t = 0$.

Convergence is an issue, but should be okay if $H^*(\mathbf{Z})$ if finitely generated in each dimension. (Maybe you just need bounded p^r-torsion in each dimension, which is a weaker condition.) Recovering information about $H^*(\mathbf{Z})$ from $H^*(\mathbf{Z}_p^\wedge)$ may also be an issue, but shouldn't be if all the groups involved are finitely generated.