Mathematics 307 Final Exam

solutions

Warning: I have not proofread these carefully. There may be typos and algebra mistakes.

1. For full credit for this question, express your answers in sigma (Σ) notation.

   (a) Fact: $\int e^{-x^2}\,dx$ cannot be expressed in terms of familiar functions. It has a Maclaurin series, though; find it. (First find the Maclaurin series for e^{-x^2}, then integrate.)

   Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then by substituting $-x^2$ for x, I get

   $$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

   Now integrate:

   $$\boxed{\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} + c}.$$

   (b) Use the definition of the Taylor series to find the Taylor series for $\sin x$ at $x_0 = \pi/2$.

   The function in question is $f(x) = \sin x$, and I know its derivatives: $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, and after that they repeat. Plug $x_0 = \pi/2$ into each of these: $f(x_0) = 1$, $f'(x_0) = 0$, $f''(x_0) = -1$, $f'''(x_0) = 0$, and after that they repeat. So the pattern is: the even ones are zero, and the odd ones alternate 1, -1, 1, -1, etc. So the Taylor series is

   $$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = 1 + \frac{0}{1!}(x-\pi/2) + \frac{-1}{2!}(x-\pi/2)^2 + \frac{0}{3!}(x-\pi/2)^3 + \cdots.$$

   After removing all of the terms that are zero (which are the odd powers of $x - \pi/2$), I get

   $$1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 - \frac{1}{6!}(x-\pi/2)^6 + \cdots.$$

   This is equal to

   $$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\pi/2)^{2n}}.$$

2. Solve the following equation using a power series about $x_0 = 0$. Find the recurrence relation; also find the first four nonzero terms in each of two linearly independent solutions. If possible, find the general term in those solutions.

   $$(1-x)y'' - y = 0.$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$. Next I plug these in: the equation becomes

$$(1-x)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0,$$

or (multiplying out the $1-x$ part)

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Now I reindex the first two sums so that the generic term involves x^n, and the result is

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=-1}^{\infty} (n+1)na_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

The $n = -2$ and $n = -1$ terms of the first sum are zero, as is the $n = -1$ term of the second sum, so I can change all of these to be sums starting at $n = 0$. So I get

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} - a_n\right] x^n = 0.$$

So for every $n \geq 0$,

$$(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} - a_n = 0,$$

and this gives the recursion relation

$$a_{n+2} = \frac{(n+1)na_{n+1}}{(n+2)(n+1)} + \frac{a_n}{(n+2)(n+1)},$$

or (slightly simplified)

$$\boxed{a_{n+2} = \frac{na_{n+1}}{n+2} + \frac{a_n}{(n+2)(n+1)}}.$$

Now I start plugging in values of n. When $n = 0$, this tells me

$$a_2 = \frac{0a_1}{2} + \frac{a_0}{2 \cdot 1} = \frac{a_0}{2}.$$

When $n = 1$, I get

$$a_3 = \frac{a_2}{3} + \frac{a_1}{6} = \frac{a_0}{6} + \frac{a_1}{6}.$$

The next few coefficients are

$$a_4 = \frac{2a_3}{4} + \frac{a_2}{12} = \frac{a_0}{12} + \frac{a_1}{12} + \frac{a_0}{24} = \frac{a_0}{8} + \frac{a_1}{12},$$

$$a_5 = \frac{3a_4}{5} + \frac{a_3}{20} = \frac{3a_0}{40} + \frac{a_1}{20} + \frac{a_0}{120} + \frac{a_1}{120} = \frac{a_0}{12} + \frac{7a_1}{120}.$$

So the general solution is

$$y = a_0 + a_1 x + \frac{a_0}{2}x^2 + (\frac{a_0}{6} + \frac{a_1}{6})x^3 + (\frac{a_0}{8} + \frac{a_1}{12})x^4 + (\frac{a_0}{12} + \frac{7a_1}{120})x^5 + \cdots .$$

I can break this up into terms involving a_0 and terms involving a_1:

$$y = a_0\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{12}x^5 + \cdots\right) + a_1\left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{7}{120}x^5 + \cdots\right).$$

So here are two linearly independent solutions:

$$\boxed{y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{12}x^5 + \cdots},$$

$$\boxed{y_2 = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{7}{120}x^5 + \cdots}.$$

I don't see a pattern, so I don't know the general term in either of these.

3. Consider this initial value problem:

$$y'' + xy' + (\cos x)y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

(a) Find $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$.
To find $y''(0)$, just plug $x = 0$ into the equation:

$$y''(0) + 0 \cdot y'(0) + (\cos 0)y(0) = 0,$$

so $y''(0) + 1 = 0$, so $\boxed{y''(0) = -1}$. To find $y'''(0)$, differentiate the original equation and then plug in $x = 0$:

$$y''' + xy'' + y' + (\cos x)y' + (-\sin x)y = 0,$$

so $\boxed{y'''(0) = 0}$. Repeat for $y^{(4)}(0)$:

$$y^{(4)} + xy''' + 2y'' + (\cos x)y'' - 2(\sin x)y' - (\cos x)y = 0.$$

So (skipping the zero terms):

$$y^{(4)}(0) + 2y''(0) + (\cos 0)y''(0) - (\cos 0)y(0) = 0,$$

which is the same as $y^{(4)}(0) - 2 - 1 - 1 = 0$, so $\boxed{y^{(4)}(0) = 4}$.

(b) What do you expect the radius of convergence for the solution y to be?
The coefficient of y'' is already 1, so I don't have to divide by anything. Both x and $\cos x$ have Taylor series (with $x_0 = 0$) with infinite radius of convergence, so I expect the same of y: the radius of convergence is $\boxed{\infty}$.

4. Find the radius of convergence for each of the following series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{2^n} x^{2n}$.

   Use the ratio test:

$$\lim_{n\to\infty} \left|\frac{(n+1)\text{st term}}{n\text{th term}}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)x^{2(n+1)}/2^{n+1}}{(-1)^n n x^{2n}/2^n}\right| = \lim_{n\to\infty} \left|\frac{(-1)(n+1)x^2}{2n}\right|$$
$$= \frac{|x^2|}{2} \lim_{n\to\infty} \left|\frac{n+1}{n}\right|.$$

   The last limit is 1. We want the whole thing to be less than 1, so $x^2/2 < 1$, so $x^2 < 2$, so $|x| < \sqrt{2}$. So the radius of convergence is $\boxed{\sqrt{2}}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!n}$.

   Use the ratio test:

$$\lim_{n\to\infty} \left|\frac{x^{n+1}/(n+1)!(n+1)}{x^n/n!n}\right| = \lim_{n\to\infty} \left|\frac{xn}{(n+1)(n+1)}\right| = |x| \lim_{n\to\infty} \left|\frac{n}{(n+1)(n+1)}\right|.$$

   This last limit is 0, so the whole thing is less than 1 no matter what x is. So the radius of convergence is $\boxed{\infty}$.

5. Solve this initial value problem:

$$ty\frac{dy}{dt} = 1, \quad y(1) = 2.$$

   This is a first order separable equation, so separate the variables:

$$y\,dy = \frac{dt}{t}.$$

   Now integrate both sides: $y^2/2 = \ln t + c$. Solve for y:

$$y = \pm\sqrt{2\ln t + c}.$$

   (Strictly speaking, that should be "$2c$," but I've renamed my constant, as usual.) Now apply the initial condition: when $t = 1$, $y = 2$: $2 = \pm\sqrt{2\ln 1 + c} = \pm\sqrt{c}$. So c should be 4, and I want the plus sign, not the minus sign:

$$\boxed{y = \sqrt{2\ln t + 4}}.$$

6. **Do not solve the problems in parts (a)–(d).** Instead, identify the type of problem ("separable," "second order linear homogeneous with constant coefficients," things like that), and tell me what method (or methods) to use to solve it ("integrating factor," "power series solution," etc.).

(a) $t^2y'' - t(t+2)y' + (t+2)y = 0$, $y_1(t) = t$.

This is a second order linear homogeneous equation (non-constant coefficients). Since we are told one solution, y_1, use reduction of order to find a second one.

(b) $\dfrac{dy}{dt} = y(y-1)(y-2)$.

This is a first order separable equation, and in particular it is an autonomous equation. If you could do the integrals, you could solve it by separating the variables and integrating. For this particular equation, it is probably better to do quantitative analysis, of the sort we did with the population growth problems in Chapter 2.

(c) $y'' + 4y = 2\tan 2t$.

This is second order linear nonhomogeneous with constant coefficients. Use the characteristic equation to find y_h, the solution to the associated homogeneous equation, and use variation of parameters to find y_p. (You can't use undetermined coefficients because the nonhomogeneous part isn't one of the right forms.)

(d) $\dfrac{dy}{dt} + \dfrac{2}{t}y = e^{2t}$.

This is a first order linear equation. Solve it using an integrating factor ($\mu(t) = e^{\int 2/t\,dt}$, etc.).

7. Find the general solution to this differential equation:

$$y'' - 4y' - 5y = -10\sin t + 2\cos t.$$

This is a second order linear nonhomogeneous equation with constant coefficients. Use the characteristic equation to find y_h: the characteristic equation is $r^2 - 4r - 5 = 0$, and this has roots 5 and -1. So $y_h = c_1e^{5t} + c_2e^{-t}$.

Use undetermined coefficients to find y_p: try $y_p = A\sin t + B\cos t$. I'm too lazy to type in the resulting algebra, but I think you should end up with $A = 1$ and $B = -1$. So the answer is

$$\boxed{y = c_1e^{5t} + c_2e^{-t} + \sin t - \cos t}.$$

8. (**Bonus**) An iron ball is hanging on a spring; there is friction present. If subject to no external forces, the position $y(t)$ of the ball at time t would satisfy this equation:

$$y'' + 2y' + 2y = 0.$$

At time 0, the ball is at its equilibrium position and has downward velocity 1. At time $t = \pi$, I turn on an electromagnet that exerts a constant downward force of magnitude 1 on the ball; at time $t = 2\pi$, I turn off the magnet. Write down the initial value problem described by this situation, and solve it.

I can't tell you how to do every problem, can I?