Mathematics 307 Exam

1. For each of the following differential equations and initial value problems: tell me what sort of equation it is (autonomous, linear first order, etc.). Then solve it if you can. If you can't solve it, or if it's really messy to solve, then instead tell me whatever qualitative information you can about the solutions. Try to present your work as clearly as possible.

   (a) $2y'' + y' - 4y = 0$.

   This is a second order linear homogeneous equation with constant coefficients. This means that I can use the characteristic equation, $2r^2 + r - 4 = 0$, to solve it. The roots of the characteristic equation are $r = \frac{-1 \pm \sqrt{1-4(2)(-4)}}{4} = \frac{-1 \pm \sqrt{33}}{4}$. Since the roots are real and distinct, the general solution is

   $$\boxed{y = c_1 e^{\frac{-1+\sqrt{33}}{4}t} + c_2 e^{\frac{-1-\sqrt{33}}{4}t}}.$$

   (b) $y' = (e^{2y} - 1)(y + 3)$.

   This is a (first order) autonomous equation. It is also separable, but one of the integrals looks pretty nasty—when you separate the variables, you get

   $$\frac{dy}{(e^{2y} - 1)(y + 3)} = dt,$$

   and I don't know, offhand, how to integrate the left side. So I'll give quantitative information instead of giving a precise solution.

   The critical points are the y-values where $y' = 0$, which occur when $e^{2y} - 1 = 0$ (that is, when $y = 0$), and when $y + 3 = 0$ (that is, when $y = -3$). y' is positive when $y > 0$, negative when $-3 < y < 0$, and positive when $y < -3$; thus $y = 0$ is an *unstable* critical point, and $y = -3$ is an *asymptotically stable* critical point.

   From this, I can sketch solutions; see the graph posted on the web site.

   (c) $y' + \frac{2}{t}y = \frac{\cos t}{t^2}$. (Assume that $t > 0$ if you need to).

   This is a first order linear equation, so I'll solve it using an integrating factor. It's already in its standard form, so the integrating factor is $\mu(t) = e^{\int 2/t \, dt} = e^{2 \ln t} = t^2$. So multiply by that:

   $$t^2 y' + 2ty = \cos t.$$

   Now integrate both sides:

   $$t^2 y = \sin t + c.$$

   Solve for y:

   $$\boxed{y = \frac{\sin t + c}{t^2}}.$$

(d) $y' = \dfrac{1-2x}{y}$, $y(1) = -2$.

This is a separable first order equation, so separate the variables and integrate:

$$y\,dy = (1-2x)dx.$$

So $\frac{1}{2}y^2 = x - x^2 + c$, so $y = \pm\sqrt{2x - 2x^2 + c}$. Plug in the initial condition: when $x = 1$, $y = -2$. Since y is negative, we'd better choose the negative square root rather than the positive one. Then we can solve for c: $c = 4$. So the solution is

$$\boxed{y = -\sqrt{2x - 2x^2 + 4}}.$$

Notice that $y = \sqrt{2x - 2x^2 + 4}$ is a solution to the differential equation, but does not satisfy the initial condition.

(e) $y' - \dfrac{1}{2}y - 2\cos t = 0$, $y(0) = -1$.

This is a first order linear equation. Put it into its standard form by moving the $2\cos t$ to the right side. Then the integrating factor is $e^{\int(-1/2)dt} = e^{-t/2}$. When you multiply by this, the equation becomes

$$e^{-t/2}y' - \frac{1}{2}e^{-t/2}y = 2e^{-t/2}\cos t.$$

Integrate both sides, using the provided table of integrals to do the right side:

$$e^{-t/2}y = 2\left(\frac{e^{-t/2}}{\frac{1}{4}+1}(-\frac{1}{2}\cos t + \sin t)\right) + c.$$

Solve for y:

$$y = \frac{8}{5}(-\frac{1}{2}\cos t + \sin t) + ce^{t/2}.$$

(Notice that the c has been multiplied by $e^{t/2}$.) Use the initial condition to solve for c: you should get $c = -1/5$. So the answer is

$$\boxed{y = -\frac{4}{5}\cos t + \frac{8}{5}\sin t - \frac{1}{5}e^{t/2}}.$$

2. State Euler's formula.

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}.$$

3. Determine whether each of the following statements is true or false. If it's true, explain why (briefly). If it's false, give a counterexample.

   (a) Suppose $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are arbitrary complex numbers. True or false: $\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) = \mathrm{Re}(z_1 z_2)$.

   This is $\boxed{\text{false}}$. Just about any pair of complex numbers you think of will give a counterexample. For instance, let $z_1 = 2 + i$ and $z_2 = 2 + 3i$. Then $\mathrm{Re}(z_1) = 2$, $\mathrm{Re}(z_2) = 2$, so $\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) = 4$. On the other hand, $z_1 z_2 = (2+i)(2+3i) = 1+8i$, so $\mathrm{Re}(z_1 z_2) = 1$. Since $4 \neq 1$, the statement is false.

   (b) Let $z = a + ib$ be an arbitrary complex number. True or false: $|\overline{z}| = |z|$.

   This is $\boxed{\text{true}}$, and there are a number of good reasons that you could give. For example: the formula for $|z|$ is $\sqrt{a^2 + b^2}$, so $|\overline{z}| = |a - ib| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$. Another good explanation: $|z|$ is equal to the "length" of z—the distance from the origin to the point z in the complex plane. Since $\overline{z}$ is the reflection of z across the real axis, the length of $\overline{z}$ will be equal to the length of z. Another good explanation: write z in polar coordinates: $z = re^{i\theta}$. Then $|z| = r$. With this notation, $\overline{z} = re^{-i\theta}$, so $|\overline{z}| = r$ as well.

4. Consider the differential equation

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

   (a) Verify that $y_1 = t^{1/2}$ and $y_2 = t^{-1}$ are solutions to this equation.

   (Just plug the two functions into the equation and show that they both work.)

   (b) Use the Wronskian to determine whether they are linearly independent.

   The Wronskian is

$$W = y_1 y_2' - y_1' y_2 = t^{1/2}(-t^{-2}) - \frac{1}{2}t^{-1/2}t^{-1} = -\frac{3}{2}t^{-3/2}.$$

   Since this is nonzero, y_1 and y_2 are $\boxed{\text{linearly independent}}$.

5. (**Bonus**) A chain four feet long starts with one foot hanging over the edge of a table. Assume there is no friction, so the chain will slide off the table. Write down a differential equation describing the motion of the chain, and solve it to find the time required for the chain to slide off the table.

   Let $y(t)$ be the length of the chain hanging off the table at time t. The force acting on the chain is due to the weight of the part of the chain which is hanging down. If the chain has total mass m, then the hanging part has weight $m\frac{y}{4}$—it's the fraction of m corresponding to the fraction of the chain that's hanging. Then Newton's second law $F = ma$ says that

$$mg\frac{y}{4} = ma,$$

   and since $a = y''$, I get this differential equation:

$$my'' = mg\frac{y}{4}, \quad \text{or} \quad my'' - \frac{mg}{4}y = 0, \quad \text{or} \quad \boxed{y'' - \frac{g}{4}y = 0}.$$

This is a second order linear homogeneous equation, with constant coefficients, so I can solve it using the characteristic polynomial $r^2 - g/4 = 0$. This has roots $r = \pm\sqrt{g}/2$, so the general solution is

$$y = c_1 e^{\sqrt{g}t/2} + c_2 e^{-\sqrt{g}t/2}.$$

The initial conditions are $y(0) = 1$ (there is one foot of chain hanging off at the start) and $y'(0) = 0$ (the chain starts at rest). I can use these to solve for c_1 and c_2: I get $c_1 = c_2 = 1/2$. So the equation for y is

$$y = \frac{1}{2}e^{\sqrt{g}t/2} + \frac{1}{2}e^{-\sqrt{g}t/2}.$$

I want to know the value of t when $y = 4$ (when the entire chain has slid off the table). So I let $y = 4$ and solve for t.

From this point, it's just algebra, but the algebra may not be obvious. First I'll solve for $e^{\sqrt{g}t/2}$. Let $u = e^{\sqrt{g}t/2}$ for now, in which case $e^{-\sqrt{g}t/2} = 1/u$. So when $y = 4$, the equation is

$$4 = \frac{1}{2}u + \frac{1}{2}u^{-1}.$$

Multiply by u:

$$\frac{1}{2}u^2 - 4u + \frac{1}{2} = 0.$$

This is a quadratic equation in u, so it has roots

$$u = 4 \pm \sqrt{16 - 1} = 4 \pm \sqrt{15}.$$

Now, $u = e^{\sqrt{g}t/2} = 4 \pm \sqrt{15}$, so take logs of both sides: $\sqrt{g}t/2 = \ln(4 \pm \sqrt{15})$, so

$$t = \frac{2\ln(4 \pm \sqrt{15})}{\sqrt{g}}.$$

$4 - \sqrt{15}$ is less than 1, so its log is negative, so t ends up being negative. So the answer must be the one with the plus sign:

$$\boxed{t = \frac{2\ln(4 + \sqrt{15})}{\sqrt{g}} = \frac{2\ln(4 + \sqrt{15})}{\sqrt{32}} = \frac{\ln(4 + \sqrt{15})}{2\sqrt{2}} = 0.73}.$$