MATHEMATICS 307 **COMPLEX NUMBERS**

The introduction of complex numbers in the 16th century was a natural step in a sequence of extensions of the positive integers, starting with the introduction of negative numbers (to solve equations of the form $x + a = 0$), the introduction of rational numbers (to solve equations like $qx + p = 0$, p, q integers) and the introduction of irrational numbers (to solve equations like $x^2 - 2 = 0$). The introduction of $i = \sqrt{-1}$ made it possible to solve the equation $x^2 + 1 = 0$, in fact, *any* quadratic equation. Pleasantly enough, one does not need any further extensions to solve an arbitrary polynomial equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0;$$

such an equation always has n roots (possibly complex and possibly repeated). These notes will present one way of defining complex numbers and familiarize you with some of their properties.

The Complex Plane. A *complex number* z is given by a pair of real numbers x and y and is written in the form $z = x + iy$, where i satisfies $i^2 = -1$. The complex numbers may be represented as points in the plane (sometimes called the Argand diagram). The real number 1 is represented by the point $(1, 0)$, and the complex number i is represented by the point $(0, 1)$. The x-axis is called the "real axis", and the y-axis is called the "imaginary axis". For example, the complex numbers $3 + 4i$ and $3 - 4i$ are illustrated in FIG 1A.

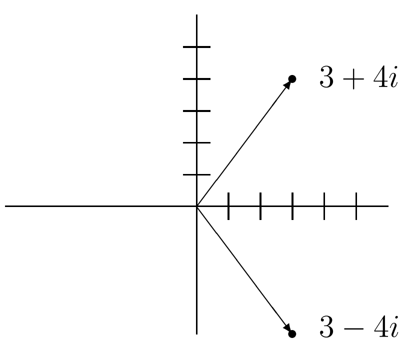

FIG 1A

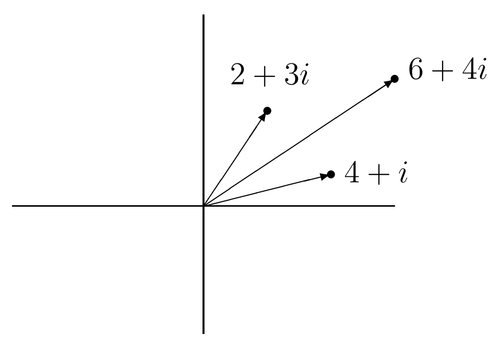

FIG 1B

Complex numbers are added in a natural way: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{1}$$

FIG 1B illustrates the addition $(4 + i) + (2 + 3i) = (6 + 4i)$. Multiplication is given by

$$z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Note that the product behaves exactly like the product of any two algebraic expressions, keeping in mind that $i^2 = -1$. Thus,

$$(2 + i)(-2 + 4i) = 2(-2) + 8i - 2i + 4i^2 = -8 + 6i$$

We call x the *real part* of z and y the *imaginary part*, and we write $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. (**Remember**: $\operatorname{Im} z$ is a *real* number.) The term "imaginary" is an historical holdover; it took mathematicians some time to accept the fact that i (for "imaginary", naturally) was a perfectly good mathematical object. Electrical engineers (who make heavy use of complex numbers) reserve the letter i to denote electric current and they use j for $\sqrt{-1}$.

There is only one way we can have $z_1 = z_2$, namely, if $x_1 = x_2$ and $y_1 = y_2$. An equivalent statement (one that is important to keep in mind) is that $z = 0$ if and only if $\operatorname{Re} z = 0$ and $\operatorname{Im} z = 0$. If a is a real number and $z = x+iy$ is complex, then $az = ax+iay$ (which is exactly what we would get from the multiplication rule above if z_2 were of the form $z_2 = a + i0$). Division is more complicated (although we will show later that the *polar representation* of complex numbers makes it easy). To find z_1/z_2 it suffices to find $1/z_2$ and then multiply by z_1. The rule for finding the reciprocal of $z = x + iy$ is given by:

$$(2) \qquad \frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

The expression $x - iy$ appears so often and is so useful that it is given a name. It is called the *complex conjugate* of $z = x+iy$ and a shorthand notation for it is $\overline{z}$; that is, if $z = x+iy$, then $\overline{z} = x - iy$. For example, $\overline{3+4i} = 3 - 4i$, as illustrated in the FIG 1A . Note that $\overline{\overline{z}} = z$ and $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$. Exercise (3b) is to show that $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$. Another important quantity associated with a given complex number z is its *modulus*

$$|z| = (z\overline{z})^{1/2} = \sqrt{x^2+y^2} = \left((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2\right)^{1/2}$$

Note that $|z|$ is a *real* number. For example, $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. This leads to the inequality

$$(3) \qquad \operatorname{Re} z \leq |\operatorname{Re} z| = \sqrt{(\operatorname{Re} z)^2} \leq \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = |z|$$

Similarly, $\operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|$. Another inequality concerning the modulus is the important *triangle inequality*

$$(4) \qquad |z_1 + z_2| \leq |z_1| + |z_2|$$

To prove this, it suffices to show that the square of the left side is less than the square of the right, so we look at

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) = z_1\overline{z}_1 + 2\operatorname{Re} z_1\overline{z}_2 + z_2\overline{z}_2.$$

(The last equality uses Exercise 3 applied to $z_1\overline{z}_2$.) Using the fact (from (3)) that $2\operatorname{Re} z_1\overline{z}_2 \leq 2|z_1\overline{z}_2| = 2|z_1||z_2|$, we get

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2,$$

which is what we wanted. A useful consequence of the triangle inequality is the following:

$$(5) \qquad ||z_1| - |z_2|| \leq |z_1 - z_2|$$

Exercises I.

(1) Prove that the product of $z = x + iy$ and the expression in (2) (above) equals 1.

(2) Verify each of the following:

(2a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)=-2i$ **(2b)** $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}=-\dfrac{2}{5}$

(2c) $\dfrac{5}{(1-i)(2-i)(3-i)}=\dfrac{1}{2}i$ **(2d)** $(1-i)^4=-4$

(3a) Prove that $z+\overline{z}=2\text{Re}z$ and that z is a real number if and only if $\overline{z}=z$.

(3b) Prove that $\overline{z_1z_2}=\overline{z}_1\overline{z}_2$.

(4a) Prove that $|z_1z_2|=|z_1||z_2|$ (Hint: Use (3b).)

(4b) Prove the inequality in (5). (Hint: By (4), $|z_1|=|(z_1-z_2)+(z_2)|\leq|z_1-z_2|+|z_2|$.)

5. Find all complex numbers $z=x+iy$ such that $z^2=1+i$.

$$\text{Ans}: z=\pm\left[\sqrt{\frac{\sqrt{2}+1}{2}}+i\sqrt{\frac{\sqrt{2}-1}{2}}\right].$$

Polar Representation of Complex Numbers. Recall that the plane has polar coordinates as well as rectangular coordinates. The relation between the rectangular coordinates (x,y) and the polar coordinates (r,θ) is

$$x=r\cos\theta \qquad \text{and} \qquad y=r\sin\theta$$

$$r=\sqrt{x^2+y^2} \qquad \text{and} \qquad \theta=\arctan\frac{y}{x}$$

(If $z=0$, then $r=0$ and θ can be anything.)

Thus, for the complex number $z=x+iy$, we can write

$$z=r(\cos\theta+i\sin\theta).$$

There is another way to rewrite this expression for z. The power series representation for e^x in powers of x is given by

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$$

For any complex number z, we *define* e^z by the power series:

$$e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots+\frac{z^n}{n!}+\ldots$$

In particular,

$$\begin{aligned} e^{i\theta} &= 1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\ldots+\frac{(i\theta)^n}{n!}+\ldots \\ &= 1+i\theta-\frac{\theta^2}{2!}-i\frac{\theta^3}{3!}+\frac{\theta^4}{4!}+\ldots \end{aligned}$$

Recall the power series for $\cos\theta$ and $\sin\theta$:

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + (-1)^n \frac{\theta^{2n}}{(2n)!} + \ldots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \ldots$$

Thus (the power series for $e^{i\theta}$) = (the power series for $\cos\theta$) + $i \cdot$ (the power series for $\sin\theta$) This is the Euler Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

For example,

$$e^{i\pi/2} = i, \quad e^{\pi i} = -1 \quad \text{and} \quad e^{2\pi i} = +1$$

Given $z = x + iy$, then z can be written in the form $z = re^{i\theta}$, where

(6) $$r = \sqrt{x^2 + y^2} = |z| \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}$$

For example the complex number $z = 8 + 6i$ may also be written as $10e^{i\theta}$, where $\theta = \arctan(.75) = .6435\ldots$ radians. This is illustrated in Fig 2.

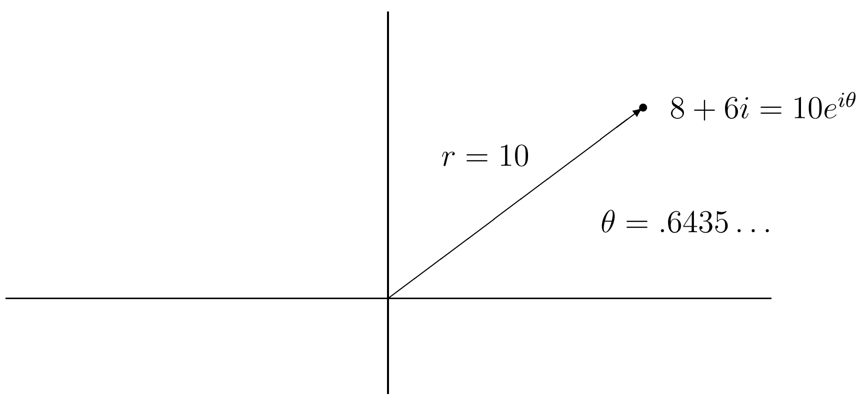

Fig 2

If $z = -4 + 4i$, then $r = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ and $\theta = 3\pi/4$, therefore $z = 4\sqrt{2}e^{3\pi i/4}$. Any angle which differs from $3\pi/4$ by an integer multiple of 2π will give us the same complex number. Thus, $-4+4i$ can also be written as $4\sqrt{2}e^{11\pi i/4}$ or as $4\sqrt{2}e^{-5\pi i/4}$. In general, if $z = re^{i\theta}$, then we also have $z = re^{i(\theta+2\pi k)}$, $k = 0, \pm1, \pm2, \ldots$. Moreover, there is ambiguity in equation (6) about the inverse tangent which can (and *must*) be resolved by looking at the signs of x and y, respectively, in order to determine the quadrant in which θ lies. If $x = 0$, then the formula for θ makes no sense, but $x = 0$ simply means that z lies on the imaginary axis and so θ must be $\pi/2$ or $3\pi/2$ (depending on whether y is positive or negative).

The conditions for equality of two complex numbers using polar coordinates are not quite as simple as they were for rectangular coordinates. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 = z_2 \text{ if and only if } r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2\pi k,\ k = 0, \pm1, \pm2, \ldots.$$

Despite this, the polar representation is very useful when it comes to multiplication:

$$\text{(7)} \qquad \text{if } z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}, \quad \text{then} \quad z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

To see why this is true, write $z_1 z_2 = re^{i\theta}$, so that $r = |z_1 z_2| = |z_1||z_2| = r_1 r_2$ (the next-to-last equality uses Ex (4a)). It remains to show that $\theta = \theta_1 + \theta_2$, that is, that $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, (this is Exercise (7a)). For example, let

$$z_1 = 2 + i = \sqrt{5} e^{i\theta_1}, \qquad \theta_1 = \tan^{-1}(\frac{1}{2}) = 0.464\ldots$$

$$z_2 = -2 + 4i = \sqrt{20} e^{i\theta_2}, \qquad \theta_2 = \tan^{-1}(-2) = -1.1071 \cdots + \pi = 2.0344\ldots$$

Then $z_3 = z_1 z_2$, where:

$$z_3 = -8 + 6i = \sqrt{100} e^{i\theta_3} \qquad \theta_3 = \tan^{-1}(-\frac{3}{4}) = 2.498\ldots$$

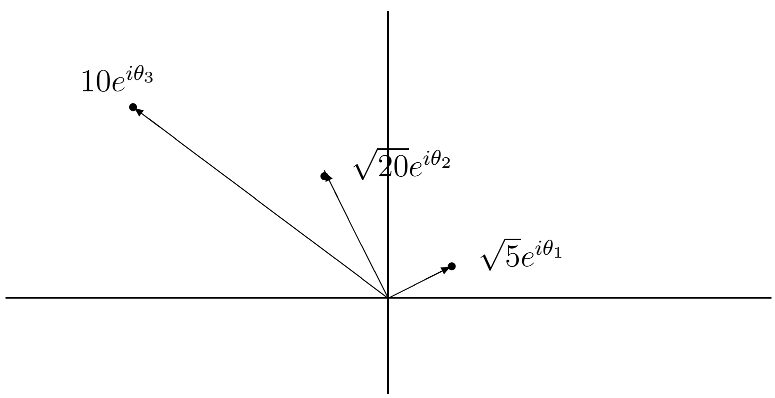

Fig 3

Applying (7) to $z_1 = z_2 = -4 + 4i = 4\sqrt{2} e^{\frac{3}{4}\pi i}$ (our earlier example), we get

$$(4 + 4i)^2 = (4\sqrt{2} e^{\frac{3}{4}\pi i})^2 = 32 e^{\frac{3}{2}\pi i} = -32i.$$

By an easy induction argument, the formula in (7) can be used to prove that for any positive integer n

$$\text{If} \quad z = re^{i\theta}, \quad \text{then} \quad z^n = r^n e^{in\theta}.$$

This makes it easy to solve equations like $z^3 = 1$. Indeed, writing the unknown number z as $re^{i\theta}$, we have $r^3 e^{i3\theta} = 1 \equiv e^{0i}$, hence $r^3 = 1$ (so $r = 1$) and $3\theta = 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. It follows that $\theta = 2k\pi/3$, $k = 0, \pm 1, \pm 2, \ldots$. There are only three distinct complex numbers of the form $e^{2k\pi i/3}$, namely $e^0 = 1$, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. The following figure illustrates $z = 8i = 8e^{i\pi/2}$ and its three cube roots $z_1 = 2e^{i\pi/6}$, $z_2 = 2e^{5i\pi/6}$, $z_3 = 2e^{9i\pi/6} = 2e^{3\pi i/2}$.

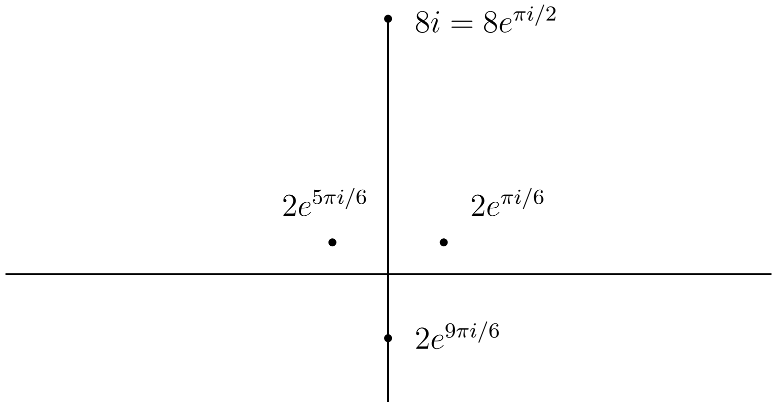

FIG 4

¿From the fact that $(e^{i\theta})^n = e^{in\theta}$ we obtain De Moivre's formula:

$$(cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

By expanding on the left and equating real and imaginary parts, this leads to trigonometric identities which can be used to express $\cos n\theta$ and $\sin n\theta$ as a sum of terms of the form $(\cos\theta)^j(\sin\theta)^k$. For example, taking $n = 2$ one gets $\cos 2\theta = \cos^2\theta - \sin^2\theta$. For $n = 3$ one gets $\cos 3\theta = \cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos^2\theta$.

Exercises II

(6) Let $z_1 = 3i$ and $z_2 = 2 - 2i$

(6a) Plot the points $z_1 + z_2$, $z_1 - z_2$ and $\overline{z_2}$.

(6b) Compute $|z_1 + z_2|$ and $|z_1 - z_2|$.

(6c) Express z_1 and z_2 in polar form.

(7a) Prove that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.

Hint: This is the same as showing that $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$.

(7b) Use (a) to show that $(e^{i\theta})^{-1} = e^{-i\theta}$, that is, $e^{-i\theta}e^{i\theta} = 1$.

(8) Let $z_1 = 6e^{i\pi/3}$ and $z_2 = 2e^{-i\pi/6}$. Plot z_1, z_2, z_1z_2 and z_1/z_2.

(9) Find all complex numbers z which satisfy $z^3 = -1$. Ans: There are three distinct such numbers: $e^{\pi i/3}, e^{\pi i} \equiv -1$ and $e^{5\pi i/3}$.

(10) Find all complex numbers $z = re^{i\theta}$ such that $z^2 = \sqrt{2}e^{i\pi/4}$. Ans: $z = 2^{\frac{1}{4}}e^{\pi i/8}$, $2^{\frac{1}{4}}e^{9\pi i/8}$. Compare with Exercise I.5; this is the polar form of the latter.