Math 307: Infinite Series
(1997–1998)

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Contents

Chapter 1. Introduction 1

Chapter 2. Taylor Polynomials 5
1. The Tangent Line Approximation 5
2. Higher order approximations 6
3. Applications of Taylor's Formula. 9

Chapter 3. Taylor Series. 15
1. Definitions 15
2. Examples 17
3. Geometric Series. 18
4. Two more important infinite series 21
5. Functions defined by power series 26

Answers to Exercises 29

CHAPTER 1

Introduction

These notes are concerned with the following problem:

> *Given a function f of the real variable x, compute $f(x_0)$, the value of f at $x = x_0$.*

Of course, sometimes the answer is rather obvious. For example, if

$$f(x) = 5 + 2x + 7x^2 + 2x^3$$

and we wish to compute the number $f(3)$ then we can do so by elementary arithmetic:

$$f(3) = 5 + 2(3) + 7(3)^2 + 2(3)^3 = 128\,.$$

In fact, the value of any polynomial can be computed using only addition and multiplication, provided only that we know its coefficients. However, polynomials (and ratios of polynomials) are the only functions whose values are easily computable. In particular, it is not obvious how to go about computing, *without a calculator*, the values of such expressions as

$$e^2, \quad \ln(2), \quad \sin(1), \quad \tan^{-1}(1)\,.$$

Of course, with a modern scientific calculator these computations are easy—but then other questions arise: How does the calculator perform such calculations? What errors are involved in the calculator's calculations?

In these notes you will learn how to do such calculations.[1]

The idea behind the method is best explained by example. Let's begin with a polynomial itself, say

$$f(x) = (1 + x)^{15}\,.$$

This *is* a polynomial and so (with a little patience) you can compute any value of f you choose. For example, to compute $f(0.01) = (1.01)^{15}$ you could just sit down and multiply 1.01 by itself 15 times. If you do this you will find that (to 4 decimal places)

$$(1.01)^{15} = 1.1610\,.$$

However, there is another way to get the same result (and with a lot less work). Recall that the binomial theorem shows that $(1 + x)^{15}$ can be written in the form

$$(1 + x)^{15} = 1 + 15x + \frac{15 \cdot 14}{1 \cdot 2}x^2 + \frac{15 \cdot 14 \cdot 13}{1 \cdot 2 \cdot 3}x^3 + \cdots + x^{15}\,.$$

[1]Modern calculators generally use more sophisticated methods than the ones you will learn. However, they have quite a bit in common with the techniques introduced here.

Suppose that you compute 1.01^{15} by first setting $x = 0.01$ and then summing the first few terms in the above expression. Here's what you get:

$$\begin{aligned}
&1 + 15(0.01) &&= 1.15 \\
&1 + 15(0.01) + \tfrac{15\cdot 14}{1\cdot 2}(0.01)^2 &&= 1.1605 \\
&1 + 15(0.01) + \tfrac{15\cdot 14}{1\cdot 2}(0.01)^2 + \tfrac{15\cdot 14\cdot 13}{1\cdot 2\cdot 3}(0.01)^3 &&= 1.160955 = 1.1610 \text{ (to 4 decimal places)} \\
&\quad \vdots
\end{aligned}$$

Thus if you only need to know 1.01^{15} to 4 places, you don't have to compute the sum of all terms; summing the first 4 terms is sufficient.

In Chapter 3 you will see that the binomial theorem also works for non-integral powers. For example,

$$\sqrt{1+x} = 1 + (1/2)x + \frac{(1/2)(1/2-1)}{1\cdot 2}x^2 + \frac{(1/2)(1/2-1)(1/2-2)}{1\cdot 2\cdot 3}x^3 + \dots .$$

Unlike the case of integer powers, this expression is *not* a polynomial, but instead involves summing an infinite number of terms! It is an example of an **infinite series**. Suppose that you want to compute $\sqrt{1.01}$. This time you *cannot* compute the number exactly— all you can do is to approximate it as accurately as you wish. My calculator (which is supposedly accurate to 10 places) produced the value

$$\sqrt{1.01} = 1.00498756\cdots = 1.0050 \text{ (to 4 decimal places)} .$$

Let's see what the binomial theorem gives:

$$\begin{aligned}
&1 + (1/2)(0.01) &&= 1.0050 \\
&1 + (1/2)(0.01) + \tfrac{(1/2)(1/2-1)}{1\cdot 2}(0.01)^2 &&= 1.00498750
\end{aligned}$$

Note that to get 4 place accuracy you only need the first 2 terms of the sum and that using the first 3 terms gives 6 place accuracy. In fact, the more terms used the more accurate the approximation of $\sqrt{1.01}$ will be. But in all cases we are using polynomials, which are easily computable, to approximate non-polynomials, which are difficult or impossible to compute.

The example just given leads to the following questions:

- Given a function f, is there an "infinite series" which represents f in the sense that summing a sufficient number of terms determines the value of f with as much accuracy as desired?
- Suppose that there is such a representation. How many terms must be summed to attain a given accuracy?

The answers to these and related questions constitute the theory of power series representations of functions, called Taylor series, the main topic of these notes.

To the Student: This set of notes has been prepared specifically for use in Math 307. No unnecessary material has been included—you are responsible for *all* material, including the problems at the end of sections. Answers to problems are provided at the end of the notes; but an answer to a problem should only be looked at *after* you have spent at least 30 minutes trying to do it.

To the Instructor: These notes are intended to be an *informal* introduction to Taylor's Theorem with Remainder and to power series representations of functions. They are *not* intended as a rigorous introduction to numerical infinite series—especially not to the *theory* behind infinite series (these topics are addressed at length in Math 325).

CHAPTER 2

Taylor Polynomials

In this chapter and the following we show that many of the common functions of mathematics can be represented by their "Taylor series". This chapter introduces the Taylor polynomials F_n of a function f. The following chapter defines of the Taylor series of a function f as the limit of its Taylor polynomials.

The idea behind our approach is easily explained. As we noted in the introduction, computing the value of a polynomial is a fairly straightforward (though sometimes messy) procedure. Therefore, we will be able to approximate the value of a function f if we can find a polynomial function which closely approximates f. Then, rather than computing the value of f directly, we can instead compute the value of the polynomial. The result will only be an approximation of the value of f, but we will show that in many cases we can choose the polynomial so that it approximates the value of f as closely as we wish.

1. The Tangent Line Approximation

Actually, you are all familiar with the simplest such approximation—it is the "tangent line approximation" of elementary calculus. Suppose that f has a derivative at $x = 0$. Then the **tangent line approximation** (also called the **linear approximation**) of f at $x = x_0$ is the linear function

$$F_1(x) = f(x_0) + f'(x_0)\,(x - x_0)\,.$$

Notice that $F_1(x_0) = f(x_0)$ and $F_1'(x_0) = f'(x_0)$.

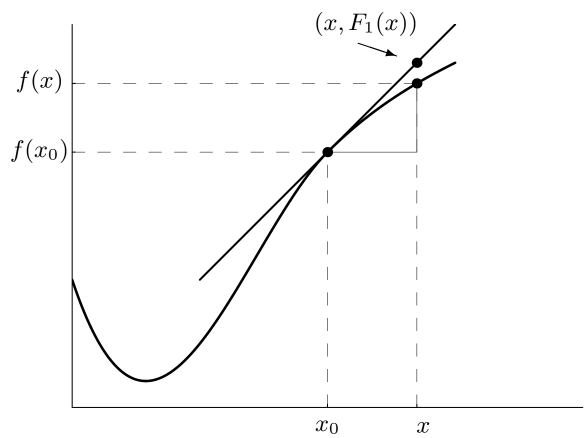

Figure 1: The Tangent Line approximation.

Another way to view the linear approximation is to look for a function of the form

$$F_1(x) = a_0 + a_1(x - x_0)$$

satisfying the two conditions,

$$F_1(x_0) = f(x_0) \quad \text{and } F_1'(x_0) = f'(x_0)\,.$$

Noting that $F_1(x_0) = a_0$ and $F_1'(x_0) = a_1$ gives the formulas

$$a_0 = f(x_0) \qquad a_1 = f'(x_0)\,.$$

Example 1.1. In many instances the tangent line approximation is sufficiently accurate to be of use. The linear approximation of $f(x) = e^x$ at $x = 0$ is given by

$$F_1(x) = 1 + x\,.$$

The graphs of $y = e^x$ and $y = F_1(x)$ are shown below:

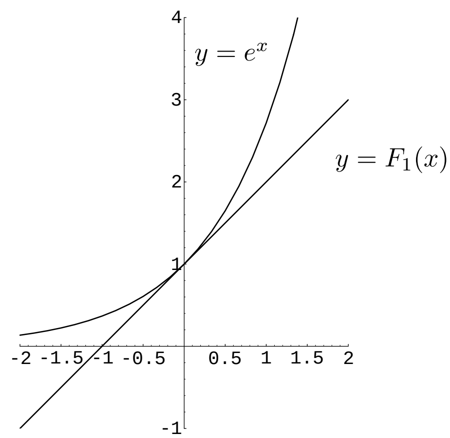

Figure 2: The Tangent Line approximation to $y = e^x$ at $x = 0$.

For small values of x this is a reasonable approximation. For example

$$e^{0.05} = 1.05127109637\ldots \text{ and } F_1(0.05) = 1.0500$$

and so

$$e^{0.05} - F_1(0.05) = 0.001271\cdots < 0.005\,.$$

Thus, the tangent line approximation to e^x at $x = 0$ enables us to approximate the value of $e^{0.05}$ to *two decimal places.*[1]

2. Higher order approximations

As you can see from the figure, however, the tangent line approximation to $f(x) = e^x$ is not very good for values of x not near 0. Suppose we want to approximate $e^{0.05}$ more closely or to approximate e^x for larger values of x. How should we proceed? One way to get more accuracy is to try replace the tangent line approximation by a polynomial of higher degree. Let's search for a second degree polynomial

$$F_2(x) = a_0 + a_1 x + a_2 x^2$$

[1]Given numbers A and B we say that B approximates A to m decimal places means that the difference $|A - B|$ is at most most $5.0 \times 10^{-(m+1)}$. For example, since $|e^{0.05} - F_1(0.05)| = 0.00127\cdots < 5.0 \times 10^{-3}$, we have approximated $e^{0.05}$ to two decimal places. Note that this does not necessarily mean that the decimal expansions of $f(x)$ and $F_n(x)$ agree for the first m decimal places: Consider $f(x) = 0.199997$ and $F_n(x) = 0.200001$.

approximating $f(x) = e^x$. We choose the coefficients a_0, a_1 and a_2 so that the following equalities hold:

$$f(0) = F_2(0), \quad f'(0) = F_2'(0), \quad F_2''(0) = f''(0)\,.$$

Note that

$$F_2(0) = a_0, \quad F_2'(0) = a_1, \quad F_2''(0) = 2a_2$$

and

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1\,.$$

Solving for the coefficients gives

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2}$$

and we arrive at the formula

$$F_2(x) = 1 + x + \frac{x^2}{2}\,.$$

Observe that,

$$F_2(0.05) = 1.05125000\,.$$

Thus

$$|e^{0.05} - F_2(0.05)| = 0.000021\cdots < 0.00005$$

and $F_2(0.05)$ approximates $e^{0.05}$ to 4 decimal places. The following figure shows the graph of $y = e^x$, together with the approximating polynomials $F_1(x)$ and $F_2(x)$:

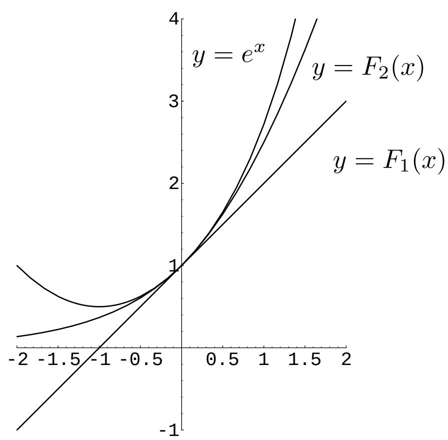

Figure 3: The linear and quadratic approximations to $y = e^x$ at $x = 0$.

To get a better approximation all we have to do is use a polynomial of higher dergree. As an easy exercise, show that

$$F_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

is a 3^{rd} degree polynomial approximating e^x, whose first 3 derivatives agree with those of e^x for $x = 0$. How closely does $F_3(0.05)$ approximate $e^{0.05}$? It is worthwhile to draw graphs of $y = e^x$ and $y = F_3(x)$ similar to those shown if Figures 2 and 3.

The general case. In the general case, we want to approximate a function $f(x)$ for values of x near a fixed value x_0 by a polynomaial $F_n(x)$ of degree n. We choose $F_n(x)$ so that its first n derivatives agree with those of $f(x)$ when $x = x_0$. That is, we require that the identities

$$F_n^{(k)}(x_0) = f^{(k)}(x_0) \text{ for } k = 0, 1, \ldots, n$$

are all satisfied.

Because $F_n(x)$ is a polynomial of degree n, we can write it in the form.

$$F_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n,$$

It is not difficult to see that the formulas

$$F_n(x_0) = a_0, \quad F_n'(x_0) = a_1, \quad F_n''(x_0) = 2!a_2, \quad \ldots, F_n^{(k)}(x_0) = k!a_k\,, \ldots\; F_n^{(n)}(x_0) = n!a_n$$

hold. Solving for the coefficients a_k yields the important identity,

$$a_k = \frac{f^{(k)}(x_0)}{k!}, \quad k = 0, 1, 2, \ldots, n.$$

Definition 2.1. Let f be a function defined on an interval $(x_0 - A, x_0 + A)$, having derivatives of all orders. For any integer $n \geq 0$, we define the n^{th} **Taylor polynomial of** f at $x = x_0$ to be the polynomial

$$F_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n$$

where $a_0 = f(x_0)$, $a_1 = f'(x_0)$, $a_2 = f''(x_0)/2!$, $\ldots$, $a_n = f^{(n)}(x_0)/n!$. Thus,

$$F_n(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2!}\,(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n\,.$$

The difference $f(x) - F_n(x)$ is called the **remainder at the n^{th} stage**, and is denoted by $R_n(x)$.

Remark 2.2. Note that we have, by definition,

$$f(x) - F_n(x) = R_n(x)$$

so that

$$f(x) = F_n(x) + R_n(x).$$

The remainder $R_n(x)$ should be thought of as an error term. When it is sufficiently small the approximation

$$f(x) \approx F_n(x)$$

is a good one. Our hope then is that we can make the remainder $R_n(x)$ arbitrarily small by choosing n sufficiently large. In other words, we hope that

$$R_n(x) \to 0 \text{ as } n \to \infty \qquad \text{for } x \text{ in } (-A, A)$$

or equivalently that

$$\lim_{n\to\infty} F_n(x) = f(x) \qquad \text{for } x \text{ in } (-A, A)\,.$$

In such cases we can use the Taylor polynomials F_n to approximate the function f to as many decimal places as we like.

The next theorem gives a formula for the remainder term $R_n(x)$ which will prove useful is estimating its size.

Theorem 2.3 (Taylor's Formula with Remainder). *Let f be a function having derivatives of all orders in an interval $(x_0 - A, x_0 + A)$ and let F_n be its n^{th} Taylor polynomial at $x = x_0$ and R_n be the remainder at the n^{th} stage, so that*

$$f(x) = F_n(x) + R_n(x).$$

Then there exists a number c_x between x_0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}$$

We do not prove this theorem in these notes. However, we shall learn to use its conclusion in a number of ways.

The special case $n = 0$ is called the **Mean Value Theorem**:

$$f(x) = f(x_0) + f'(c_x)(x - x_0) \quad \text{or} \quad \frac{f(x) - f(x_0)}{x - x_0} = f'(c_x)\,.$$

3. Applications of Taylor's Formula.

The method of using Taylor's formula with remainder to estimate the accuracy of the approximation of a function by its n^{th} Taylor polynomial is best illustrated by an example.

Example 3.1. We saw in Example 1.1 that

$$\begin{aligned} e^x &= F_1(x) + R_1(x) \\ &= 1 + x + R_1(x)\,. \end{aligned}$$

How good is the linear approximation

$$e^x \approx 1 + x\,?$$

We expect it to be good for small values of x but bad for large values. Let's restrict ourselves in advance to the interval $0 \le x \le 1$. Taylor's formula furnishes us with a method for estimating the size of the error term $R_1(x)$ for x in this interval.

Recall from elementary calculus that the function e^x has infinitely many derivatives and that all derivatives are equal to e^x. Taylor's formula applies then to give the formula

$$R_1(x) = \frac{e^{c_x}}{2}x^2$$

where c_x is a number between 0 and x which depends on x. Unfortunately Taylor's formula does not tell us how to find c_x; and so it does not tell us how to find $R_1(x)$. Moreover, we should not expect to be able to find $R_1(x)$. For suppose we knew $R_1(x)$. Of course we know $F_1(x)$ (it's a polynomial), so we could then compute $e^x = F_1(x) + R_1(x)$. But we started out with the assumption that we did not know how to compute e^x.

Thus, the best we can hope for is a reasonable estimate of the size of $R_1(x)$. Suppose that x is a positive number, then because c_x is between 0 and x and $f(x) = e^x$ is an increasing function of x, it is certainly the case that e^{c_x} is no larger than e^x; and, because x is no larger than 1, it is certainly the case that e^x is no bigger than e^1. Thus:

$$R_1(x) = \frac{e^{c_x}}{2}x^2 \le \frac{e^x}{2}x^2 \le \frac{e^1}{2}x^2\,.$$

Recalling that $e \approx 2.7 < 3$ we replace e by 3. The price we pay is a slightly exaggerated estimate for the size of $R_1(x)$ (but we avoid having to know the precise value of e):

$$R_1(x) \le \frac{e}{2}x^2 < 1.5x^2\,.$$

This formula gives us a ballpark estimate of the error involved in the approximation

$$e^x \approx 1 + x \text{ for } 0 \le x \le 1$$

For example, $e^{0.05} \approx 1.05$ with an error of at most $(1.5)(0.05)^2 = 0.0038$.

We next wish to discuss a method that is commonly used to make tables of values of functions. As previously mentioned, the error we commit by using the value $F_n(x)$ as an approximation of the value of $f(x)$ is $f(x) - F_n(x)$ i.e. $R_n(x)$. If we have a good estimate for R_n, we can tell how large n should be in order to achieve a specified accuracy in our approximation. Let us look at two examples.

Example 3.2. Suppose we want to make a table of values of $f(x) = e^x$ for values of x between 0 and 1, accurate to four decimal places when rounded off. This means that the error between the value we calculate and the correct value must not exceed 5×10^{-5} (why?). We use the Taylor polynomial F_n to do this.

Recall that $f^{(k)}(x) = e^x$ for all k. Hence $f^{(k)}(0) = 1$ for all k and we see that

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!} \quad k = 0, 1, 2, \dots .$$

Hence the n^{th} Taylor polynomial of e^x is

$$F_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} .$$

We have

$$\begin{aligned} e^x = f(x) &= F_n(x) + R_n(x) \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x) . \end{aligned}$$

If we take $F_n(x)$ to be an approximate value for e^x, we commit an error whose size is $R_n(x)$. If we choose n so large that this error is less than 5×10^{-5}, then our approximation will be correct to 4 decimal places. Now by Taylor's formula with remainder, we know that there is a number c_x between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1} .$$

Since $f^{(n+1)}(c_x) = e^{c_x}$,

$$R_n(x) = \frac{e^{c_x}}{(n+1)!} x^{n+1} .$$

We do not know how big c_x is but we *know an upper bound* for it. In fact since c_x is between 0 and x and x is between 0 and 1, we know that c_x is no bigger than 1. Also since x is no bigger than 1, x^{n+1} is also no bigger than 1. Thus

$$|R_n(x)| \le \frac{e^1}{(n+1)!} \cdot 1^{(n+1)} = \frac{e}{(n+1)!} .$$

Now, we know that $e \le 3$ so finally we get the estimate

$$|R_n(x)| \le \frac{3}{(n+1)!} \qquad \text{for all } x \text{ in } [0, 1] .$$

Therefore, if we choose n large enough so that the right side of this inequality is less than 5×10^{-5}, we will be ensuring that the remainder term, and hence the error in our approximation, is also less then 5×10^{-5}. To find how large n must be we can make a table.

$n =$	1	2	3	4	5	6	7	8
$3/(n+1)! =$	1.5	.5	.125	.025	.004	.0006	.000075	.0000083

Thus we see that a suitable value for n is $n = 8$ so the approximation

$$e^x \approx 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^8}{8!}$$

is accurate to 4 decimal places for computing e^x for x between 0 and 1. This formula can be used to compute the value of e to 4 decimal places:

$$e = e^1 \approx 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{8!}\,.$$

Computing the right hand side to 4 places gives $e \approx 2.7183$.

EXAMPLE 3.3. Suppose we want to construct a table of values of $\cos x$ for x between 0 and $\pi/2$.

Here we use again Taylor's formula. Let $f(x) = \cos x$. Then $f'(x) = -\sin x$, $f''(x) = -\cos x$. Proceeding this way we see that each derivative of odd order equals either $-\sin x$ or $\sin x$, and each derivative of even order equals either $\cos x$ or $-\cos x$.

In fact we see easily that

$$\begin{aligned} f^{(2k)}(x) &= (-1)^k \cos x \quad k = 0,1,2,\ldots \\ f^{(2k+1)}(x) &= (-1)^{k+1} \sin x \quad k = 0,1,2,\ldots\,. \end{aligned}$$

It follows that

$$\begin{aligned} f^{(2k)}(0) &= (-1)^k \quad k = 0,1,2,\ldots \\ f^{(2k+1)}(0) &= 0 \qquad k = 0,1,2,\ldots\,. \end{aligned}$$

In other words all the odd order derivatives vanish at 0. For the Taylor coefficients we have

$$\begin{aligned} a_{2k} &= \tfrac{f^{(2k)}(0)}{(2k)!} = \tfrac{(-1)^k}{(2k)!} \quad k = 0,1,2,\ldots \\ a_{2k+1} &= 0 \qquad\qquad\quad k = 0,1,2,\ldots\,. \end{aligned}$$

Thus, $a_1 = a_3 = a_5 = \cdots = 0$, while $a_0 = 1$, $a_2 = -\frac{1}{2!}$, $a_4 = \frac{1}{4!}$, $a_6 = -\frac{1}{6!}$ and so on. We find that the n^{th} Taylor polynomial for $\cos x$ has only even powers of x present in it. We have

$$F_{2m}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!}\,.$$

Suppose we use the $2m^{\text{th}}$ Taylor polynomial $F_{2m}(x)$ to approximate $\cos x$. Notice that since the odd-degree terms are zero, $F_{2m}(x)$ is the same as $F_{2m+1}(x)$. So let us put $n = 2m+1$ and use the n^{th} Taylor polynomial $F_n(x)$ which is in this case

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \pm \frac{x^{2m}}{(2m)!}$$

to approximate $\cos x$. The error we commit is

$$R_n(x) = \frac{f^{(n+1)}(c_x)x^{n+1}}{(n+1)!}\,,$$

where c_x is between 0 and x, hence between 0 and $\pi/2$. Again, since $n = 2m+1$, we have $n+1 = 2m+2$. So $f^{(n+1)}(x)$ is an even order derivative of $\cos x$, hence $f^{(n+1)}(x) = \pm\cos x$.

So $f^{(n+1)}(c_x) = \cos c_x$, thus $|f^{(n+1)}(c_x)| \leq 1$. Since x is between 0 and $\pi/2$, x^{n+1} is no bigger than $(\frac{\pi}{2})^{n+1}$. So we get the estimate

$$|R_n(x)| \leq \frac{1 \cdot (\frac{\pi}{2})^{(n+1)}}{(n+1)!}, \quad x \text{ in } (0, \pi/2).$$

Now $\pi \approx 3.14 \leq 4$ so $\pi/2 \leq 2$. Hence,

$$|R_n(x)| \leq \frac{2^{n+1}}{(n+1)!}.$$

To ensure, say, 3 place accuracy for example, we need to make the error less that 5×10^{-4}. Again we make a table:

n	1	3	5	7	9	11
$2^{n+1}/(n+1)!$	2.0	.6667	.0889	.0063	.0003	.000086

We see that $n = 9$ will do the trick. Since $n = 2m + 1$, the Taylor polynomial

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

will approximate $\cos x$ correctly to 3 decimal places for all x in $(0, \pi/2)$.

The figure below graphically illustrates that the Taylor polynomials F_2, F_4, F_6, ..., F_{10} do indeed give successively better approximations of the cosine function:

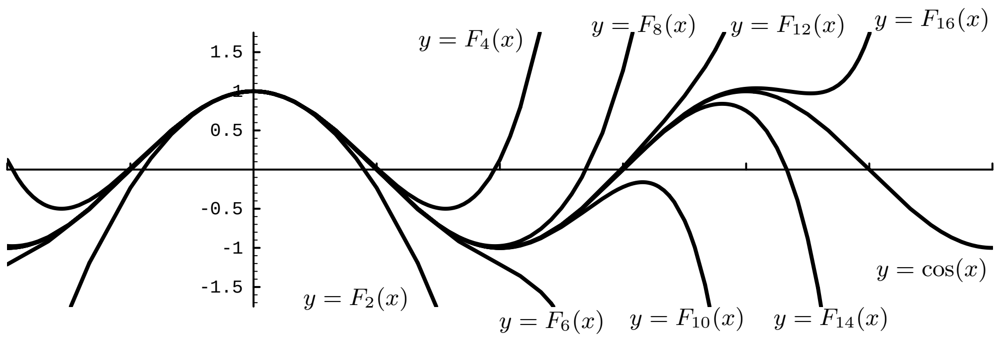

Figure 4: Several Taylor polynomials approximating $y = \cos(x)$.

Exercise Set 1.

(1) Consider the function $f(x) = e^{-x}$. Show that for $0 \le x$ the inequality
$$|R_n(x)| \le \frac{x^{n+1}}{(n+1)!}$$
holds. Use this inequality to approximate e^{-1} to 4 decimal places.

(2) Show that the 2^{nd} Taylor polynomial of the function $f(x) = \sqrt{1+x}$ is $1 + x/2 - x^2/8$. Then use Taylor's formula to estimate the error in the approximation
$$\sqrt{1.5} \approx 1 + \frac{0.5}{2} - \frac{(0.5)^2}{8}$$

(3) Show that for x between 0 and 1 the error in the approximation
$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$$
is at most $\dfrac{3x^2}{8(1-x)^{5/2}}$.

(4) In Newtonian physics the kinetic energy ("energy of motion") of a body of mass m (in kilograms) and velocity v (in meters/sec) is given by the formula $KE = (1/2)mv^2$. On the other hand, in relativity theory, the kinetic energy is given by the formula
$$\overline{KE} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$
where $c =$ speed of light $= 3 \times 10^8$ m/sec.
   (a) Let $u = (v/c)$ (velocity measured as a fraction of the speed of light) and let $y = \overline{KE}/mc^2$. Graph y as a function of u by plotting the points corresponding to $v = c/10,\ 2c/10,\ \ldots,\ 9c/10$.
   (b) Use the result of Problem 3 with $x = (v/c)^2$ to obtain an upper bound on the error in the approximation
$$\overline{KE} \approx KE\,.$$
   (c) Since in our everyday world v is small relative to the speed of light, in most practical situations the above approximation is quite good. Use the value $c = 3 \times 10^8$ m/sec together with the upper bound you just obtained to obtain an upper bound on the percent error in using the Newtonian instead of the relativistic kinetic energy (this percent error is $\frac{\overline{KE}-KE}{\overline{KE}} \times 100\%$) when:
   (i) $v = 100$ km/hr, (ii) $v = 20000$ km/hr, (iii) $v = c/2$.

(5) Consider the function $f(x) = \sin(x)$.
   (a) Find a formula for the $n = (2m+1)$-st Taylor polynomial of $f(x) = \sin(x)$. Explain why $F_{2m+1}(x) = F_{2m+2}(x)$.
   (b) Proceeding as in Example 3.3 above, show that the remainder R_{2m+2} satisfies the inequality
$$|R_{2m+2}(x)| \le \frac{|x|^{2m+3}}{(2m+3)!}\,.$$
   (c) Find a Taylor polynomial which can be used to approximate $\sin(x)$ to 5 decimal places for angles between $-\pi/2$ and $+\pi/2$ radians.

CHAPTER 3

Taylor Series.

1. Definitions

Let f be a function defined on an interval $(x_0 - A, x_0 + A)$ having derivatives of all orders and denote the n^{th} Taylor polynomial of f by

$$F_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n\,,$$

where

$$a_k = \frac{f^{(k)}(x_0)}{k!}\,, \quad k = 0, 1, 2, \ldots, n\,.$$

Recall that we can express f in the form

$$f(x) = F_n(x) + R_n(x)\,.$$

Examination of examples 3.2 and 3.3 above show that it can happen that as n approaches ∞ the remainder at the n^{th} stage approaches 0. That is,

$$\lim_{n\to\infty} (f(x) - F_n(x)) = \lim_{n\to\infty} R_n(x) = 0\,.$$

In such cases, it is possible to make the approximation

$$f(x) \approx F_n(x)$$

as accurate as we wish (we simply have to choose n big enough). An equivalent way of writing the condition $\lim_{n\to\infty} R_n(x) = 0$ is

$$\lim_{n\to\infty} F_n(x) = f(x)$$

Using "sigma notation" we can write the Taylor polynomials F_n in the form

$$\begin{aligned} F_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \\ &= f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2!}\,(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n\,. \end{aligned}$$

Using sigma notation the condition $F_n(x) \to f(x)$ as $n \to \infty$ can be rewritten in the form

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = f(x)$$

It would seem natural then to write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

or, alternately,

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2!}\,(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \dots$$

where the expression $+\dots$ after $\frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ indicates the process of "infinite summation".

1.1. **Notation.** A word concerning notation is perhaps in order here. It is a matter of convenience which of the notations

$$a_0 + a_1(x - x_0) + \dots + a_n(x - x_0)^n + \dots$$

and

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k$$

we use. Both expressions mean the same thing. The second notation has the advantage of being more compact; while the first is often easier to read. At times we will be a little sloppy and use even shorter notation such as

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

or even $\sum a_k(x - x_0)^k$.

It should also be observed that the letter k is the sigma notation is a "dummy variable" and we can replace it by any symbol we choose. Thus,

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k = \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{m=0}^{\infty} a_m(x - x_0)^m\,.$$

We formalize the above discussion in the next definition.

Definition 1.2. Let f be a function defined on the interval $(x_0 - A, x_0 + A)$ having derivatives of all orders. The **Taylor series** of f at $x = x_0$ is the expression

$$\sum_{k=0}^{\infty} a_k(x - x_0)^k \text{ where } a_k = \frac{f^{(k)}(x_0)}{k!}\,, \quad k = 0, 1, 2, \dots$$

In the special case $x_0 = 0$, the series assumes the simpler form

$$\sum_{k=0}^{\infty} a_k\, x^k \text{ where } a_k = \frac{f^{(k)}(0)}{k!}\,, \quad k = 0, 1, 2, \dots$$

This series is called the **Maclaurin series** of f.

We say that the Taylor series **converges to** $f(x)$ if the equation

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k\,(x - x_0)^k = f(x)$$

is satisfied. If the Taylor series converges to $f(x)$ for all x in the interval $(x_0 - A, x_0 + A)$, then we say that the series **converges to** f **on the interval** $(x_0 - A, x_0 + A)$.

This is the same as saying that

$$\left| f(x) - \sum_{k=0}^{n} a_k (x-x_0)^k \right| \to 0 \quad \text{as } n \to \infty,$$

which, of course, is the same as saying that for x in $(x_0 - A, x_0 + A)$, the absolute value of the remainder $R_n(x)$ tends to 0 as $n \to \infty$. Now, this is true for some functions and their Taylor series, while for others it is not true. The easiest way to show that it *is*, is to show that $|R_n(x)|$ is less than some expression (in n, or in n and x) which is itself known to converge to 0. Some such expressions are obvious; for instance, it is clear that $\frac{1}{n} \to 0$ or that $\frac{1}{n!} \to 0$ as $n \to \infty$. Here are two useful but less obvious examples:

Example 1.3.

(a) If $|x - x_0| < 1$, then $|x - x_0|^n \to 0$ as $n \to \infty$.

(b) For *any* x, $\dfrac{|x-x_0|^n}{n!} \to 0$ as $n \to \infty$.

We will assume (a) without proof. To convince you that (b) is true, observe that there is certainly an integer m such that the inequality $|x - x_0| < m$ holds. Then if $n > 2m$, $\dfrac{|x-x_0|}{n} < \dfrac{1}{2}$. Now let $A = |x-x_0|^{2m}/(2m)!$ and suppose that n is any integer larger than $2m$. If follows that

$$\begin{aligned} 0 \quad \le \quad & \frac{|x-x_0|^n}{n!} = \frac{|x-x_0|^{2m}}{(2m)!} \cdot \frac{|x-x_0|^{n-2m}}{(2m+1)(2m+2)\dots(n)} \\ = \quad & A\left(\frac{|x-x_0|}{(2m+1)}\right)\left(\frac{|x-x_0|}{(2m+2)}\right)\dots\left(\frac{|x-x_0|}{n}\right) < A\left(\frac{1}{2}\right)^{n-2m} \end{aligned}$$

But $(1/2)^{n-2m} \to 0$ as $n \to \infty$, so $|x-x_0|^n/n! \to 0$ as $n \to \infty$.

2. Examples

Some examples will put a little meat on the idea of Taylor Series.

The exponential function. Let $f(x) = e^x$. It is defined and has derivatives of all orders on the interval $(-A, A) = (-\infty, +\infty)$. From Example 3.2 we easily see that its Taylor series at $x_0 = 0$ (i.e. its Maclaurin series) is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \dots$$

However, *just because we have computed the Maclaurin series of e^x does not mean that it converges to e^x—we have to* prove *it!*

Fortunately, this is not so difficult to do. We have only to show that

$$|R_n(x)| \to 0 \text{ as } n \to \infty \text{ for all } x\,.$$

Taylor's formula (see 2.3) gives

$$|R_n(x)| = \frac{e^{c_x}}{(n+1)!}|x|^{n+1}$$

where c_x is a number between 0 and x. There are two cases to consider: $x \ge 0$ and $x < 0$.

Start by assuming $x \geq 0$. Because c_x is between 0 and x we have $e^{c_x} \leq e^x$ so

$$0 \leq |R_n(x)| \leq e^x \frac{|x|^{n+1}}{(n+1)!}$$

By Example 1.3(b), $e^x \frac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$. So $R_n(x) \to 0$ as $n \to \infty$ for $x > 0$.

Now assume $x < 0$. Observe that $e^{c_x} \leq 1$ (because c_x is at most 0), so

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty\,,$$

and, therefore, $R_n(x) \to 0$ as $n \to \infty$ for $x < 0$.

Thus, for every real number x (whether positive or negative) the remainder term $R_n(x)$ approaches 0 as $n \to \infty$ and the proof is complete.

We have proved the important identity

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

or, using sigma notation,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.$$

The cosine function. Let $f(x) = \cos x$. The computations in Example 3.3 show that the Maclaurin series of $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\,.$$

We now use Taylor's formula to show that the Maclaurin series of $\cos x$ converges to $\cos x$ for all x. Choose a number x. Then Taylor's formula gives

$$|R_n(x)| = \left| f^{(n+1)}(c_x) \right| \frac{|x|^{n+1}}{(n+1)!}\,,$$

where c_x is a number between 0 and x. Note that $f^{(n+1)}(c_x)$ is either $\pm \sin c_x$ or $\pm \cos c_x$ for any positive integer n. In any case $|f^{(n+1)}(c_x)| \leq 1$ for all x and we may conclude,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}\,.$$

The right hand side of this inequality approaches 0 as $n \to \infty$. Again we conclude that $R_n(x) \to 0$ as $n \to \infty$. This being true for any x, we see that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\,.$$

3. Geometric Series.

Even though a function f may have derivatives of all orders, and therefore a Taylor series, it is not always the case that its Taylor series converges to the function for all values of x. Consider, by way of example, the function $f(x) = \dfrac{1}{(1-x)}$. To find the Maclaurin series of f let's start by computing its first few derivatives:

$$f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = \frac{1 \cdot 2}{(1-x)^3}, \quad f'''(x) = \frac{3!}{(1-x)^4}, \ldots\,,$$

from which we conclude that
$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}\,.$$
It follows that $a_k = f^{(k)}(0)/k! = 1$ for all k and that the Taylor series of f is
$$1 + x + x^2 + x^3 + x^4 + \ldots \text{ or, equivalently, } \sum_{k=0}^{\infty} x^k\,.$$
This series is a famous and useful one; it is called the **Geometric Series**. Using a simple algebraic identity, we can actually find $R_n(x)$ in this case explicitly.

We are going to discuss the convergence of the geometric series $1 + x + x^2 + \ldots$ by explicitly calculating its n^{th} **partial sum** $s_n = 1 + x + \cdots + x^n$, and then analyzing its behavior as $n \to \infty$. We claim that

$$1 + x + x^2 + \cdots + x^n = \begin{cases} n+1 & \text{if } x = 1 \\ \dfrac{1 - x^{n+1}}{1-x} = \dfrac{1}{1-x} - \dfrac{x^{n+1}}{1-x} & \text{if } x \neq 1. \end{cases} \tag{3.1}$$

PROOF. To prove Formula (3.1) denote the left side by s_n. If $x = 1$, it is obvious that $s_n = n$. So assume $x \neq 1$. We have
$$s_n = 1 + x + \cdots + x^n$$
so, multiplying both sides by x, we get
$$xs_n = x + x^2 + \cdots + x^{n+1}\,.$$
Subtracting the second equation from the first gives
$$(1-x)s_n = 1 - x^{n+1}\,,$$
because the terms $x, x^2, x^3, \ldots, x^n$ drop out due to cancellation. Now we divide by $1 - x$ (recall that $x \neq 1$, so this is permissible), and obtain
$$s_n = \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}.$$
□

Notice that we can thus interpret formula (3.1) in terms of the remainder at the n^{th} stage. First note that we can rewrite it in the form
$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \frac{x^{n+1}}{1-x}\,.$$
On the other hand, we know that
$$\frac{1}{1-x} = 1 + x + \cdots + x^n + R_n(x)\,.$$
Thus
$$R_n(x) = \frac{x^{n+1}}{1-x} \text{ for } x \neq 1\,.$$
We can now easily determine the values of x for which the geometric series converges. If $x = 1, s_n = n + 1$ and clearly the series does not converge. If $x \neq 1$, we consider the expression
$$s_n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}.$$

The first term on the right does not depend on n. The behaviour of the 2^{nd} term depends on the behaviour of x^n as $n \to \infty$. Recall (Example 1.3(a)) that if $|x| < 1$ then x^n approaches 0 as n approaches ∞. While, if $|x| > 1$, x^n does not approach zero as n. We have therefore proved the following theorem.

THEOREM 3.2. *The geometric series*

$$1 + x + x^2 + \dots$$

converges to $\dfrac{1}{1-x}$ *if* $|x| < 1$. *It does not converge if* $|x| \geq 1$.

This theorem is the cornerstone of the theory of infinite series.

A generalization of the geometric series. There is an important, though simple, generalization of the geometric series:

$$a + ar + ar^2 + ar^3 + \dots .$$

This series is also called a **geometric series**. Notice that the ratio of successive terms is just r, itself; i.e.

$$\frac{ar^{n+1}}{ar^n} = r\,.$$

For this reason r is called the **ratio** of the geometric series. Notice also, that the partial sums can be written

$$a + ar + ar^2 + \cdots + ar^n = a\left(1 + r + r^2 + \cdots + r^n\right) = a\left(\frac{1}{1-r} - \frac{r^{n+1}}{1-r}\right)$$

This simple observation shows that Theorem 3.2 can be then rephrased as follows:

> *The geometric series* $a + ar + ar^2 + \dots$ *converges to* $\dfrac{a}{1-r}$ *if the absolute value of its ratio is less than* 1. *If* $|r| \geq 1$ *then the geometric series does not converge.*

We now give a number of examples and applications of the geometric series.

EXAMPLE 3.3. The series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots +$$

is a geometric series with the common ratio $r = \frac{1}{2}$. By Theorem 3.2, we see that it converges to $\frac{1}{1-\frac{1}{2}}$, which equals 2.

EXAMPLE 3.4. The series

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} + \cdots + (-1)^n \frac{1}{3^n} + \dots$$

is a geometric series with common ratio $-\frac{1}{3}$. Since $|-\frac{1}{3}| = \frac{1}{3}$ which is < 1, Theorem 3.2 implies that this series converges. Its sum is $\frac{1}{1-(-\frac{1}{3})} = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$.

EXAMPLE 3.5. The series

$$1 + 2 + 4 + \cdots + 2^n + \dots$$

is a geometric series with common ratio 2. Since $2 > 1$, Theorem 3.2 applies to show that the series does not converge.

Example 3.6. This example deals with the present value of delayed payments during a period of inflation. Assume a constant net annual rate of inflation of r, and suppose that a payment of A dollars is made at the end of a year. The **present value** of A is the quantity $P = (1+r)^{-1}A$. The reasoning behind this terminology is the following: If P dollars were to be invested at the beginning of the year at a net interest rate of r, then at the end of the year it would have increased to a value of

$$(1+r)P = (1+r)\frac{A}{(1+r)} = A\,.$$

For example a payment of \$100 one year from now at an annual net inflation rate of 5% is worth only $\$(1.05)^{-1}100 = \$95.23$ today. The reason for this is that \$95.23 invested at 5% will be worth $\$(1.05)95.23 = \$100$ at the end of a year.

Extrapolating this idea, one deduces that a payment of A dollars k years from now has a present value of

$$(1+r)^{-k}A \text{ dollars.}$$

(Why?)

Suppose that you win a lottery which will pay \$10,000 each year forever (after you die your descendants continue to receive payments). Assuming a constant rate of inflation of 5%, what is the present value of this prize if payments start immediately?

The present value of the total of *all* payments is

$$\$10000 + (1+0.05)^{-1}\$10000 + (1+0.05)^{-2}\$10000 + \dots\,.$$

(Why?) This can be rewritten in the form

$$\$10000\{1 + (1+0.05)^{-1} + (1+0.05)^{-2} + \dots\}\,.$$

The term is braces is the sum of a geometric series with common ratio $(1+0.05)^{-1} < 1$. Thus the present value of the prize is

$$\$10000\left\{\frac{1}{1-(1+0.05)^{-1}}\right\} = \$10000(1+1/0.05) = \$210,000\,.$$

The general formula is easy to derive: If the inflation rate is r and with annual payments of A dollars the total value of the prize is

$$A\left\{1 + (1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3}\dots\right\} = \frac{A}{1-(1+r)^{-1}} = \left(1+\frac{1}{r}\right)A$$

dollars.

4. Two more important infinite series

The natural logarithm. Let $f(x) = \ln(1+x)$ in the interval $(-1,1)$. Then $f'(x) = 1/(1+x) = (1+x)^{-1}$. $f''(x) = -1(1+x)^{-2}$, $f'''(x) = (-2)(-1)(1+x)^{-3} = (-1)^2 2!(1+x)^{-3}$, $f^{(4)}(x) = (-3)(-2)(-1)(1+x)^{-4} = (-1)^3 3!(1+x)^{-4}$, and so on. Proceeding in this way, we get $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$. It follows that

$$f^{(n)}(0) = (-1)^{n-1}(n-1)! \quad n = 1,2,3,\dots\,;$$

hence,

$$\begin{aligned} a_n &= \frac{f^{(n)}(0)}{n!} \\ &= \frac{(-1)^{n-1}(n-1)!}{n!} \\ &= \frac{(-1)^{n-1}}{n} \quad n = 1, 2, \dots . \end{aligned}$$

Thus the Taylor series is

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots .$$

We now consider the problem of showing that the Taylor series converges to $\ln(1+x)$ for all x in $(-1, 1]$. The n^{th} Taylor polynomial of $\ln(1+x)$ is

$$F_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n}.$$

Now

$$f^{(n+1)}(x) = (-1)^n n! (1+x)^{-(n+1)},$$

so $f^{(n+1)}(c_x)$ equals $(-1)^n n!(1+c_x)^{-n-1}$, and hence

$$\begin{aligned} R_n(x) &= \frac{(-1)^n n! (1+c_x)^{-(n+1)}}{(n+1)!} \cdot x^{n+1} \\ &= \frac{(-1)^n (1+c_x)^{-(n+1)}}{n+1} \cdot x^{n+1} . \end{aligned}$$

Supposing first that x is positive, we see that since c_x is between 0 and x, we must have $c_x > 0$, so $1 + c_x > 1$. It follows that $(1+c_x)^{-(n+1)} < 1$. Hence

$$|R_n(x)| \le \frac{|x|^{n+1}}{n+1} .$$

Now if $|x| \le 1$, the right side $\to 0$ as $n \to \infty$. Hence we see that the Taylor polynomials of $f(x)$ converge to f. We have proved that for $0 \le x \le 1$,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots ;\ 0 \le x \le 1 .$$

The situation is more complicated when x is negative and the argument will not be given here. However, the same formula for $\ln(1+x)$ is valid for all x in $(-1, 1]$.

The binomial theorem. Let $f(x) = (1+x)^p$, where p is a real number. Then $f'(x) = p(1+x)^{p-1}$, $f''(x) = p(p-1)(1+x)^{p-2}, \dots$ and in general, $f^{(n)}(x) = p(p-1)(p-2)\dots(p-n+1)(1+x)^{p-n}$ for $n = 1, 2, \dots$. Therefore we find that $f^{(n)}(0) = p(p-1)(p-2)\dots(p-n+1)$, and hence

$$\begin{aligned} a_n &= \frac{f^{(n)}(0)}{n!} \\ &= \frac{p(p-1)\dots(p-n+1)}{n!} \quad n = 1, 2, 3, \dots . \end{aligned}$$

The Maclaurin series for $(1+x)^p$ is therefore

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots .$$

This series is called **Newton's binomial series** for $(1+x)^p$. When p is an integer, the series terminates because eventually there will be a factor $(p-p)$ in the product defining a_n, and so a_n will equal zero as soon as $n > p$. If p is not an integer, $a_n \neq 0$ for all n. The Taylor series of $(1+x)^p$ converges to $(1+x)^p$ for all x in $(-1, 1)$. This fact was discovered by Newton.

Exercise Set 2.

(1) Prove that the identity

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

is valid for all x.

(2) The hyperbolic sine and hyperbolic cosine are the functions defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \text{ and } \cosh(x) = \frac{e^x + e^{-x}}{2}$$

respectively. Using the identities $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$, obtain the identities

$$\begin{aligned} \sinh(x) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\ \cosh(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \end{aligned}$$

by finding the appropriate Taylor polynomial and showing that the remainders converge to 0.

(3) Show that the Maclaurin series of the function $\sqrt{1+x}$ can be written in the form

$$1 + \left(\frac{1}{2}\right) x - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) x^2 + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{3}{6}\right) x^3 - \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{3}{6}\right)\left(\frac{5}{8}\right) x^4 + \dots$$
$$+(-1)^{k+1}\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{3}{6}\right)\left(\frac{5}{8}\right)\dots\left(\frac{2k-3}{2k}\right) x^k + \dots$$

and that the Maclaurin series of the function $\dfrac{1}{\sqrt{1+x}}$ can be written in the form

$$1 - \left(\tfrac{1}{2}\right) x + \left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right) x^2 - \left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{6}\right) x^3$$
$$+ \dots + (-1)^k \left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{6}\right)\dots\left(\tfrac{2k-1}{2k}\right) x^k + \dots .$$

Note: Notice that we are not claiming that these series actually converge to the functions $\sqrt{1+x}$ and $1/\sqrt{1+x}$ for all values of x. In fact, this is *not* the case—both series converge for $|x| < 1$ and both fail to converge for $|x| > 1$.

(4) A ball is dropped from a height of $h = 4.9$ meters. Suppose that air resistance is negligible, but that the energy absorbed by the pavement causes it to bounce up to a height only 0.6 times as great each time.

(a) Find the total distance traveled up and down by the ball.

(b) Now find a formula that expresses the total distance traveled up and down by the ball in terms of the height h from which it is dropped and the fraction τ of the height it bounces each time. (In part (a), $h = 4.9\,m$ and $\tau = 0.6$.)

(c) Calculate the total time the ball in part (a) travels. (**Hint:** the time it takes to fall a distance y can be computed from the formula $y = \frac{1}{2}gt^2$, with $g = 9.8\,\mathrm{m/sec}^2$, and the time taken to bounce up to a certain height is the same as the time it takes to fall from that height.)

(d) Find a formula for the total time the ball travels in terms of the height from which it was initially dropped, the fraction τ and the acceleration of gravity g.

(5) Suppose you make an equilateral triangle of side 10 cm out of wire. You then join the midpoints of the three sides to make a second, smaller equilateral triangle; then you join the midpoints of the smaller triangle to make a still smaller one; and so on indefinitely. (a) How much wire do you need? (b) What is the sum of the areas of all of the triangles?

(6) When making prescriptions for drugs that will be taken over a prolonged period of time it is necessary to take into account the fact that the concentration of a drug in the bloodstream grows after each subsequent dose. In this problem you derive a formula in standard use by physicians.

Let c_0 be the concentration of a drug immediately after the first dose (this is proportional to the size of the dose and to the weight of the patient and is information known for all commonly used drugs). After t units of time the concentration will be given by the formula $c = c_0 e^{-rt}$ where r is a constant which depends on the drug (this is just the law of exponential decay and again the value of r is known for all commonly used drugs).

(a) Now suppose that the same dose is taken every T units of time (e.g. every 4 hours). Immediately after the second dose is taken the concentration will be $c_0 + c_0 e^{-rT}$. (Why?). Immediately after the third dose the concentration will be $c_0 + c_0 e^{-rT} + c_0 e^{-2rT}$. (Why?) What is the formula for the concentration after the n-th dose?

(b) Show that after a very long time the concentration of drug in the bloodstream immediately after a dose is taken will approach a stable value c_∞ given by the formula

$$c_\infty = \frac{c_0}{1 - e^{-rT}}\,.$$

(c) Find the value of r if the half-life of the drug in the bloodstream is 3 hours.

(d) Use the result of the previous parts of the problem to obtain a graph of the ratio c_∞/c_0 as a function of T for a drug with a half-life of 3 hours. What is the time between doses if the stable concentration is twice the initial concentration?

5. Functions defined by power series

In § 3 we found that many familiar functions in mathematics can be represented by Taylor series. For example, we derived the formulas

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots .$$

If we substitute a particular value for x, say $x = 1$, into the Taylor series of e^x we obtain the formula

$$e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots .$$

By this we mean that the sequence of numbers

$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

gets closer and close to e as n gets larger and larger. We say that "the sequence s_n approaches e as n approaches ∞ or that "the sequence s_n has a limit" and that limit is the number e. It is reasonable to define the infinite sum

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots$$

as this limit.

But it often happens that a function f is *defined* by an infinite series. For example, suppose we define a function f by the formula

$$f(x) = 1 + \frac{x}{1^2+1} + \frac{x^2}{2^2+1} + \frac{x^3}{3^2+1} + \dots .$$

By this we mean that to compute the value of f at a particular value of x, we simply substitute the value into the series and define $f(x)$ to be the limit of the infinite series. Thus, for example,

$$f(0.5) = 1 + \frac{0.5}{2} + \frac{0.5^2}{5} + \frac{0.5^3}{10} + \dots .$$

Example 5.1. Ordinary differential equations can often be solved using the ideas presented here. For example, consider the initial value problem

$$y' = y \,, y(0) = 2 \,.$$

Let $y = f(x)$ be its solution, i.e. $f'(x) = f(x)$ and $f(0) = 2$. Let's compute the Taylor series for f. Notice that we can get new equations involving higher derivatives of f by repeatedly differentiating the differential equation:

$$f'(x) = f(x), \quad f''(x) = f'(x), \quad f'''(x) = f''(x), \dots, f^{(n+1)}(x) = f^{(n)}(x), \dots$$

Setting $x = 0$ gives

$$f(0) = 2, \quad f'(0) = f(0) = 2, f''(0) = f'(0) = 2, \dots, f^{(n)}(0) = 0 \,.$$

Thus, the Taylor series for the solution $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{2}{n!}\, x^n = 2 \sum_{n=0}^{\infty} \frac{x^n}{n!} \,.$$

Of course, we know by other means that the solution is $f(x) = 2e^x$. Notice, however, that *we did not use this information!*

Exercise Set 3.

(1) Following the method of Example 5.1, find the Taylor series about $x = 0$ for a function $y = f(x)$, which solves the initial value problem $y' = 2y$, $y(0) = 1$.
(2) Following the method of Example 5.1, find the Taylor series about $x = 1$ for a function $y = f(x)$, which solves the initial value problem $y' = 2y$, $y(1) = 1$.

We continue the discussion of infinite series by moving to Section 5.1 of Boyce & DiPrima.

Answers to Exercises

Exercise Set #1.

(1) From the identity, $f^{(k)}(x) = (-1)^k e^{-x}$, and using Taylor's formula with remainder, we obtain the equation

$$|R_n(x)| = \frac{e^{-c_x}}{(n+1)!}|x|^{n+1}$$

where c_x is a number between 0 and x. By assumption, $x > 0$, therefore $0 \leq c_x \leq x$ and since $f(x)$ is a decreasing function of x it follows that $e^{-c_x} \leq 1$. Thus the inequality, $|R_n(x)| \leq x^{n+1}/(n+1)!$ holds for $x \geq 0$.

To approximate e^{-1} to 4 decimal places we must choose n so large that $|R_n(1)| \leq 0.00005$. But by the work just done we know that $|R_n(1)| \leq 1/(n+1)!$. So we need only choose n so large that $1/(n+1)! < 0.00005$. The smallest integer for which this inequality is satisfied is $n = 7$, for which $1/(n+1)! = 0.000025$. Hence,

$$e^{-1} \approx 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots - \frac{1}{7!} = 0.367857$$

which to 4 decimal places is 0.3679.

(2) From the formula $f'''(x) = \frac{3}{8(1+x)^{5/2}}$ one gets

$$R_2(x) = \frac{f'''(c_x)}{3!}x^3 = \frac{3}{48(1+c_x)^{5/2}}x^3$$

where c_x is between 0 and x. Applying this with $x = 0.5$ gives $R_2(0.5) = \frac{3}{48(1+c_x)^{5/2}}(0.5)^3 < \frac{3}{48}(0.5)^3 = 0.0078$. Thus, the error is at most 0.0078.

(3) This problem is similar to the previous problem. This time $f(x) = (1-x)^{-1/2}$ and we want to extimate $R_1(x)$, for $0 \leq x < 1$. From the equation $f''(x) = \frac{3}{4(1-x)^{5/2}}$ we have

$$|R_1(x)| = \left|\frac{f''(c_x)}{2!}x^2\right| = \frac{3}{8(1-c_x)^{5/2}}x^2 \leq \frac{3x^2}{8(1-x)^{5/2}}$$

(4) (a) clear.

(b) From the previous exercise,

$$\left|\frac{1}{\sqrt{1-x}} - \left((1 + \frac{1}{2}x\right)\right| \leq \frac{3x^2}{8(1-x)^{5/2}}\ .$$

For $x = (v/c)^2$ this becomes

$$\left|\frac{1}{\sqrt{1-v^2/c^2}} - (1 + \frac{1}{2}(v^2/c^2)\right| \leq \frac{3(v^2/c^2)^2}{8(1-v^2/c^2)^{5/2}}\ .$$

Thus

$$\begin{aligned} \left|\overline{KE} - KE\right| &= \left|\tfrac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 - \tfrac{1}{2}mv^2\right| = mc^2\left|\tfrac{1}{\sqrt{1-v^2/c^2}} - 1 - \tfrac{1}{2}v^2/c^2\right| \\ &\leq mc^2\tfrac{3v^4/c^4}{8(1-v^2/c^2)^{5/2}}\ . \end{aligned}$$

After a little algebra this becomes

$$\left|\overline{KE} - KE\right| \le \frac{3mv^4}{8c^2(1 - v^2/c^2)^{5/2}}\,.$$

(c) From the above expression we can estimate the percent error:

$$\frac{\left|\overline{KE} - KE\right|}{KE} \le \frac{3v^2}{4c^2(1 - v^2/c^2)^{5/2}}\,.$$

Setting $c = 3 \times 10^8$ m/sec gives:

$v =$	100 km/hr	2000 km/hr	c/2
% error $=$	6.43×10^{-13}	2.57×10^{-8}	38.5

(5) (a) and (b) should be clear. (c) A Taylor polynomial which approximates $\sin(x)$ to 5 decimal places for $|x| \le \pi/2$ is $x - x^3/3! + x^5/5! - x^7/7! + x^9/9!$.

Exercise Set #2.

(1) From problem 5 of Exercise Set 1 we know that $|\sin(x) - F_n(x)| \le |x|^{n+1}/(n+1)!$, where F_n is the n-th Taylor polynomial. But $x^{n+1}/(n+1)! \to 0$ as $n \to \infty$ for all x. Hence, for all x

$$\sin(x) = \lim_{n\to\infty} F_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

(2) Proceed as in the calculations of the Taylor series of cos and sin, only in the present situation there are no sign changes, hence the result.

(3) Write

$$f^{(n)}(x) = (-1)^n \frac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{2^n} (1+x)^{-\frac{(2n-1)}{2}}\,.$$

Hence,

$$|R_n(x)| = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^{n+1}(n+1)!} \frac{|x|^{n+1}}{|1 + c_x|^{\frac{2n+1}{2}}} \le \frac{|x|^{n+1}}{2n+2}\,.$$

The last inequality follows since $\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)} \frac{1}{2n+2} < \frac{1}{2n+2}$, and since $x > 0$ implies that $c_x > 0$.

(4)

(a): Total dist. $= 4.9 + 2(0.6)(4.9) + 2(0.6)^2(4.9) + 2(0.6)^3(4.9) + \ldots = 4.9 + 2(4.9)(0.6)(1 + 0.6 + (0.6)^2 + \ldots) = 4.9 + 2(4.9)(0.6)/(1-0.6) = 19.6$ meters.

(b): Total dist. $= h + 2\tau h + 2\tau^2 h + 2\tau^3 h + \ldots = h + 2\tau h(1 + \tau + \tau^2 + \ldots = h + 2\tau h \frac{1}{1-\tau}$
$= h\left(\frac{1+\tau}{1-\tau}\right)$

(c)&(d): Part (c) is part (d) with numerical values for h, τ and g. The solution to part (d) follows:

The time it takes to fall from (or reach) a height of y meters is $\sqrt{2y/g}$. Using this formula repeatedly one can compute as follows:

$$\begin{aligned}
\text{Total time} &= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2\tau h}{g}} + 2\sqrt{\frac{2\tau^2 h}{g}} + 2\sqrt{\frac{2\tau^3 h}{g}} + \ldots \\
&= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2\tau h}{g}}\left\{1 + \sqrt{\tau} + \sqrt{\tau^2} + \sqrt{\tau^3} + \ldots\right\} \\
&= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2\tau h}{g}}\frac{1}{1-\sqrt{\tau}} = \sqrt{\frac{2h}{g}}\left(\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}}\right).
\end{aligned}$$

(In part (c), where $h = 4.9$m, $\tau = 0.6$ and $g = 9.8\text{m/sec}^2$, one gets 7.87 seconds for the total time.)

(5) **(a):** $3 \cdot 10 + 3 \cdot (10/2) + 3 \cdot (10/4) + \dots = 30(1 + 1/2 + 1/4 + \dots) = 30(1/(1 - 1/2) =$ 60 cm.

(b): Area $= \frac{\sqrt{3}}{4}(10^2 + (10/2)^2 + (10/4)^2 + \dots) = \frac{\sqrt{3}}{4} \cdot 100 \cdot (1 + 1/4 + 1/16 + \dots)$ $= \frac{\sqrt{3}}{4} \cdot 100 \cdot (1/(1 - 1/4)) = \frac{100}{\sqrt{3}}$ square centimeters.

(6) **(a):** After the n-th dose the concentration is given by the formula $c_0 \sum_{k=0}^{n-1} e^{-krT}$

(b): As $n \to \infty$ the concentration approaches the value $c_0 \sum_{k=0}^{\infty} e^{-krT}$. But this is a geometric series with positive ratio e^{-rT} less than 1. Hence, the series converges to the sum $c_0\{1/(1 - e^{-rT})\}$ as we were to show.

(c): Recall that the concentration of the drug in the bloodstream t hours after the inital dose is given by the formula $c(t) = c_0 e^{-rt}$ (assuming now that only ONE dose is given). Because the half-life is 3 hours we have $c_0/2 = c_0 e^{-3r}$ so $r = \ln(2)/3$.

(d): Using the results of parts (b) and (c) we arrive at the formula

$$\frac{c_\infty}{c_0} = \frac{1}{1 - e^{-\frac{\ln 2}{3}T}} = \frac{1}{1 - 2^{-T/3}}$$

Note that $c_\infty / c_0 = 2$ when $T = 3$ hours.

Exercise Set #3.

(1) Let $y = f(x)$. Because $y' = 2y$ and $y(0) = 1$, it follows that $f(0) = 1$ and $f'(x) = 2f(x)$. So $f'(0) = 2f(0) = 2$. Differentiating the equation $f'(x) = 2f(x)$ gives $f''(x) = 2f'(x) = 4f(x)$; so $f''(0) = 4$. Continuing, we find that $f^{(k)}(x) = 2^k f(x)$, so $f^{(k)}(0) = 2^k$. It follows that the Maclaurin series for f is

$$1 + \frac{2x}{1!} + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!},$$

which is the series for e^{2x}, i.e. $y = e^{2x}$.