Worksheet # 9
Quadratic and cubic approximations
Math 124

This worksheet explores some concepts beyond the tangent line approximation. If $f(x)$ is a function, then its tangent line approximation near $x = a$ is

$$y = f(a) + f'(a)(x - a).$$

Its *quadratic approximation* is

$$y = f(a) + f'(a)(x - a) + f''(a)\frac{(x-a)^2}{2},$$

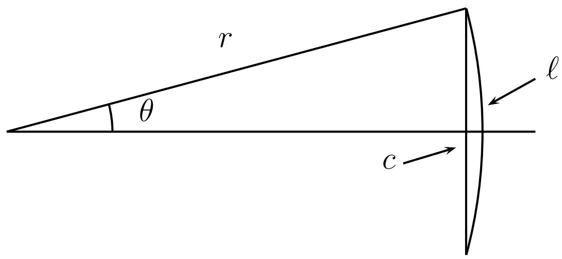
and its *cubic approximation* is

$$y = f(a) + f'(a)(x - a) + f''(a)\frac{(x-a)^2}{2} + f'''(a)\frac{(x-a)^3}{6}.$$

These approximations are only reasonable when x is near a. The tangent line approximation is the easiest to work with (because it's a line), but it's the least accurate. The quadratic approximation is better, and the cubic approximation is even better.

(a) Find the cubic approximations to $\sin x$ and $\cos x$ for $a = 0$. Compare $\sin x$ to its cubic approximation for some values of x near 0; how good does the approximation seem to be?

Consider the following picture, in which ℓ is an arc of a circle of radius r, spanned by an angle of 2θ. c is the corresponding chord of the circle. (So the top half of the arc has length $\ell/2$, and the top of the chord is $c/2$.)



(b) $r - r\cos\theta$ measures what length in the picture?

(c) Use the arc length formula and the cubic approximation for $\sin x$ to show that θ is approximately equal to $\sqrt{6\left(1 - \frac{c}{\ell}\right)}$. [So if you know c and ℓ but not r, you can find θ, approximately. To see that this is worthwhile, try solving for θ exactly.]

(d) A widget company has a factory which is exactly one mile from its store. They want to ship their widgets by train, so they try to buy a piece of train track, exactly one mile long, to connect the factory and the store. Mistakes are made, though, so the train track ends up being one foot too long. They want to use the track anyway, so they jam it into place, one end at the factory, the other end at the store. Since the track is too long, it bulges up in the middle, coincidentally into an arc of a circle. How far off the ground is the track halfway between the factory and the store?

(You will need to use the answers from (b) and (c). If you find yourself wanting to compute a sine or cosine, use its cubic approximation instead.)