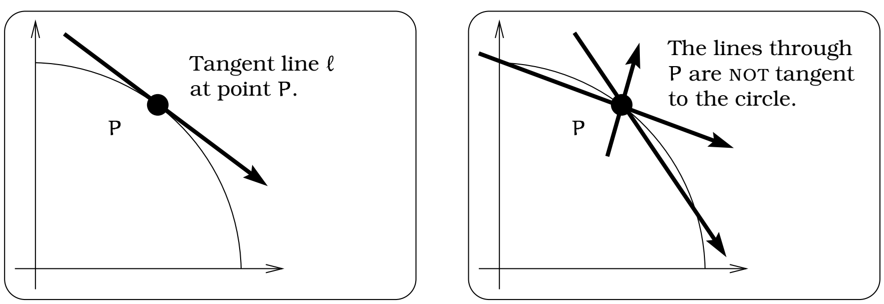
Chapter 5

Tangent Lines

Sometimes, a concept can make a lot of sense to us visually, but when we try to do some explicit calculations we are quickly humbled. We are going to illustrate this sort of thing by way of a particular example: the concept of a tangent line to a curve at a given point. The pictures below indicate what we would all agree is and is not considered to be a tangent line to the pictured circle at the point P. The motivating question is this: *What is the equation of the tangent line ℓ in Figure 5.1(a)?*



(a) Yes!

(b) No!

Figure 5.1: Is it a tangent line?

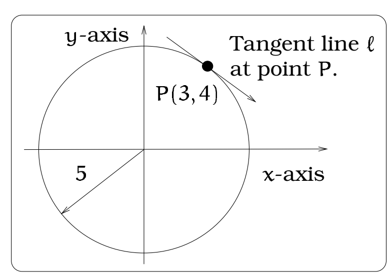


Figure 5.2: Computing the tangent line.

We will devote this section to answering this question whenever the curve is a circle or an ellipse. In these cases, we can use what we know about equations of lines, together with the quadratic formula, to calculate the tangent line to any point on a circle or an ellipse. Later, in Chapter 8, we will see the same techniques apply when the curve is a parabola. Answering the question for a general curve is the beginning of a calculus course.

To illustrate the key idea, start with the specific example of the circle of radius 5 centered at the origin. Recall, the points (x, y) that lie on this circle are precisely the ones satisfying the equation $x^2 + y^2 = 5^2 = 25$. Consequently, the point $P = (3, 4)$ will lie on this circle. What we want to do is find the equation of a line ℓ passing through P which is "tangent" to the circle. In order to do this, we need to agree on a definition for such a tangent line. We will say that a line ℓ is *tangent* to the circle at the point P, if the line and the circle only intersect at P. In other words, we are saying that the line "touches" the circle at P and only at P.

According to summary 4.3.4 on page 45, we can write down the equation of ℓ if we know two things: a point on the line and its slope m. Since we are requiring the line to pass through $P = (3, 4)$, the point-slope formula for the line gives us this equation; our job is to find m.

$$y = m(x - 3) + 4.$$

5.1 Simultaneous Equation Approach

In order to answer do this, we will use the fact that P is the unique point where both the line and circle intersect. In general, if we have a line and a circle, we can find the points of intersection by simultaneously solving the corresponding equations. So, for our case here, we would simultaneously solve the system of equations:

$$\begin{aligned} y &= m(x-3)+4 \\ 25 &= x^2 + y^2. \end{aligned}$$

To do this, we plug the first equation into the second, which eliminates the variable y, then proceed to solve for x. Let's go through that calculation in detail:

$$\begin{aligned} 25 &= x^2 + (m(x-3)+4)^2 \\ 25 &= x^2 + m^2(x-3)^2 + 8m(x-3) + 16 \\ 25 &= x^2 + m^2x^2 - 6m^2x + 9m^2 + 8mx - 24m + 16 \\ 0 &= (1+m^2)x^2 + (8m - 6m^2)x + (9m^2 - 24m - 9) \end{aligned}$$

At this stage, what we have is a quadratic equation in the variable x, where the coefficients (the things we called a, b, c in (1.4.15)) involve m:

$$\begin{aligned} a &= (1+m^2) \\ b &= (8m - 6m^2) \\ c &= (9m^2 - 24m - 9) \end{aligned}$$

We can apply the quadratic formula and obtain the values for x:

$$x = \frac{-(8m-6m^2) \pm \sqrt{(8m-6m^2)^2 - 4(1+m^2)(9m^2-24m-9)}}{2(1+m^2)}$$

What we have just done is find the x coordinate of any intersection point between the line ℓ and the circle. However, we know up front that P is the *only* intersection point. That means that the quadratic formula can only give us one value for x. That means that the term

$$\pm\sqrt{(8m-6m^2)^2-4(1+m^2)(9m^2-24m-9)}=0;$$

otherwise we would get two answers for x, which is not true! A miracle occurs and the expression for x is simply

$$x=\frac{-(8m-6m^2)}{2(1+m^2)}.$$

As noted, we are assuming $x=3$ is the only solution to the system of equations, giving the equation:

$$\begin{aligned} 3 &= \frac{-(8m-6m^2)}{2(1+m^2)} \\ 6(1+m^2) &= -(8m-6m^2) \\ 6+6m^2 &= -8m+6m^2 \\ m &= -\frac{3}{4}. \end{aligned}$$

In other words, we have shown that $y=-\frac{3}{4}(x-3)+4$ is the equation of the tangent line through P.

The procedure we have just outlined will work anytime we are given a point on a circle or an ellipse. Here, we need to make sure we agree on the definition of a tangent line to an ellipse or a circle at the point P: *A tangent line at P is a line that touches the circle or ellipse at the point P and only at the point P.* Recall, given positive constants a and b, an *ellipse* is obtained when you look at the points (x,y) in the plane that satisfy the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

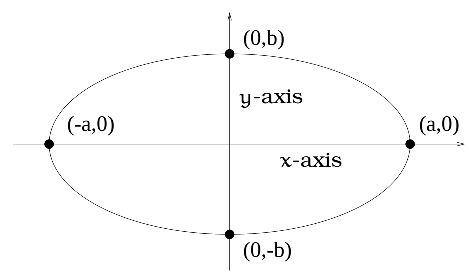


Figure 5.3: The ellipse.

The resulting graph has the pictured appearance; we have included four specific points on the ellipse which are easily determined once you are given a and b.

5.2 Radial Line Approach

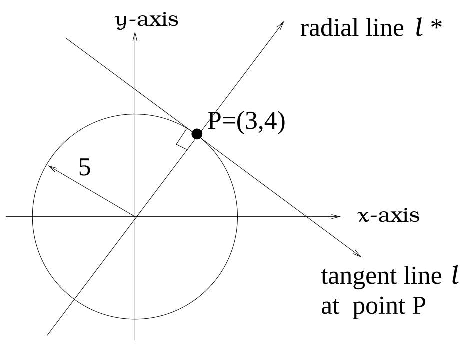


Figure 5.4: Computing the radial line.

Finally, we should point out that there is actually an alternate approach to finding the tangent line to a circle which is much easier and depends on a geometric observation. Namely, the tangent line to a circle at a point P will be perpendicular to a radial line connecting the circle center and the point P. Using the knowledge that perpendicular lines have slopes which are negative reciprocals of one another, we can quickly find the equation of the tangent line. For example, using the example above, we have this picture: The radial line ℓ^* passes through the points $(0,0)$ and $(3,4)$, so has slope $\frac{4}{3}$. That means

$$m = \text{slope } \ell = \frac{-1}{\text{slope } \ell^*} = \frac{-1}{\frac{4}{3}} = -\frac{3}{4}.$$

Certainly, the "radial line approach" is preferred over the "simultaneous equation approach". However, keep in mind, the "radial line approach" only works for circles. The "simultaneous equation approach" will work for any *conic section*: circles, ellipses, parabolas and hyperbolas.

5.3 Exercises

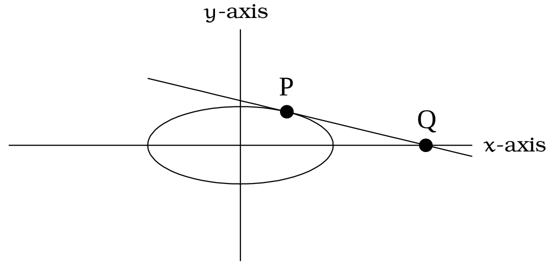
Problem 5.1. Draw the circle of radius 5 centered at the origin.

(a) Use the "radial line approach" to find the equation of the tangent line to the circle at the points $(3,-4)$, $(-4,3)$, $(-3,-4)$, $(4,-3)$.

(b) What is the equation of the tangent line to the circle at the points $(0,5)$ or $(0,-5)$?

(c) What is the equation of the tangent line to the circle at $(5,0)$ or $(-5,0)$?

Problem 5.2. If we plot the points (x,y) satisfying the equation $\frac{x^2}{4} + y^2 = 1$ the result is an *ellipse*. This ellipse is pictured below, along with a line that is tangent to the parabola at P and passes through the point $Q = (4,0)$. Use the "simulataneous equation" technique described in this section to find the equation of the tangent line and the point P where the line touches the parabola.



Problem 5.3. If we plot the points (x,y) satisfying the equation $y = -x^2 + 2$ the result is a *parabola*. This parabola is pictured below, along with a line that is tangent to the parabola at P and passes through the point $Q = (4,0)$. Use the "simulataneous equation" technique described in this section to find the equation of the tangent line and the point P where the line touches the parabola.