Mathematics 124

answers to Winter 2001 final exam

1.(a) Compute $\frac{dy}{dx}$ if $y = \dfrac{\ln(x)}{5x^3+2}$.

Solution. Use the quotient rule:

$$\frac{dy}{dx} = \frac{(5x^3+2)\frac{1}{x} - \ln(x)(15x^2)}{(5x^3+2)^2} = \boxed{\frac{5x^2 + \frac{2}{x} - 15x^2\ln(x)}{(5x^3+2)^2}}.$$

(b) Let $\Phi(t) = t\tan^{-1}(3t)$. Compute $\Phi'(t)$.

Solution. Product rule plus chain rule:

$$\Phi'(t) = \tan^{-1}(3t) + \frac{3t}{(3t)^2+1} = \boxed{\tan^{-1}(3t) + \frac{3t}{9t^2+1}}.$$

(c) Let $f(x) = \sec(e^{\sqrt{x}})$ and compute $f'(x)$.

Solution. Chain rule (several times):

$$f'(x) = \boxed{\frac{1}{2}x^{-1/2}e^{\sqrt{x}}\sec(e^{\sqrt{x}})\tan(e^{\sqrt{x}})}.$$

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2. Let $f(x) = \frac{1}{2}x^4 - 2x^3$.

(a) Determine the intervals in x where $f(x)$ is positive / negative, increasing / decreasing, concave up / concave down.

Solution. Factor $f(x)$: $f(x) = \frac{1}{2}x^4 - 2x^3 = \frac{1}{2}x^3(x-4)$. This is zero when $x = 0$ and when $x = 4$, and these are the only places $f(x)$ can change sign. When $x < 0$, $f(x)$ is positive. When $0 < x < 4$, $f(x)$ is negative. When $x > 4$, $f(x)$ is positive.

So: $f(x)$ is positive on the intervals $(-\infty, 0)$ and $(4, \infty)$. It's negative on $(0, 4)$.

$f'(x) = 2x^3 - 6x^2 = 2x^2(x-3)$. This is zero when $x = 0$ and when $x = 3$. When $x < 0$, $f'(x)$ is negative. It is also negative when $0 < x < 3$, and it is positive when $x > 3$.

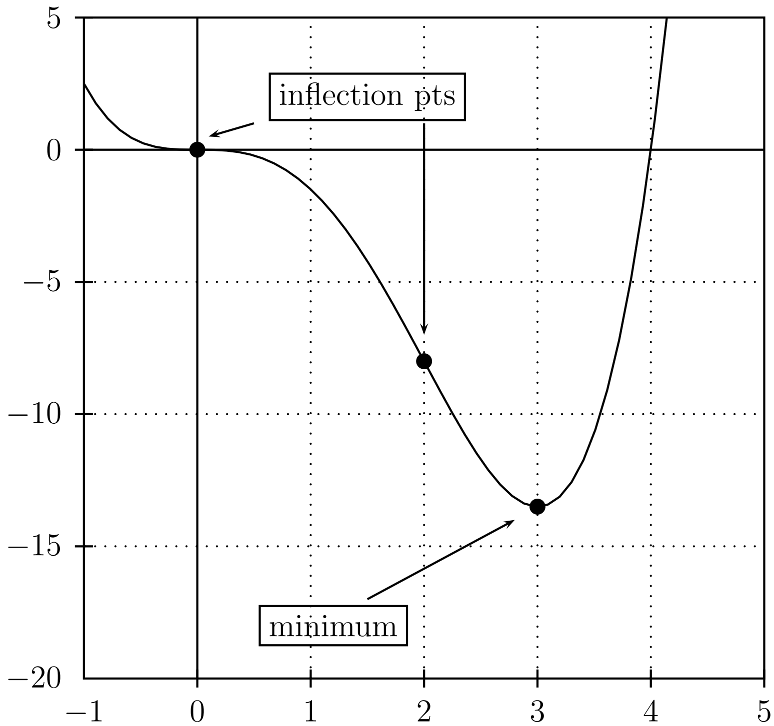
So: $f(x)$ is increasing on $(3, \infty)$. It is decreasing on $(-\infty, 0)$ and $(0, 3)$. (Thus $f(x)$ has a local minimum at $x = 3$.)

$f''(x) = 6x^2 - 12x = 6x(x-2)$. This is zero when $x = 0$, $x = 2$. When $x < 0$, $f''(x)$ is positive; it's negative when $0 < x < 2$, and it's positive when $x > 2$.

So: $f(x)$ is concave up on $(-\infty, 0)$ and $(2, \infty)$. It's concave down on $(0, 2)$. (Thus $x = 0$ and $x = 2$ are both inflection points.)

(b) Sketch a graph of $f(x)$.

Solution. First, we know $f(0) = 0$ and $f(4) = 0$. Since there is a minimum at $x = 3$, compute $f(3)$: $f(3) = -27/2 = -13\frac{1}{2}$. Since there is an inflection point at $x = 2$, compute $f(2)$: $f(2) = -8$. (And we may as well also compute $f(1) = -3/2$.)



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3. A box has a square base and open top. It is made of wood which costs \$3 a square foot. The box must hold 4 cubic feet. What dimensions minimize the box?

Solution. Let h be the height of the box, x the width of the base (so the box is $h \times x \times x$). The cost is 3 times the surface area, and we want to minimize the cost. Let C be the cost; then

$$C = 3(x^2 + 4hx),$$

because the bottom of the box is a square, $x \times x$, and there are four sides, each $h \times x$. The factor of three is because the wood costs \$ a square foot.

Also, since the volume is 4 cubic feet, we get the equation $4 = x^2 h$, so $h = 4/x^2$. Substitute this into the C equation:

$$C = 3x^2 + 12\frac{4}{x^2}x = 3x^2 + \frac{48}{x}.$$

Minimize this on the domain $x > 0$ (since the width x can't be negative, because it's a length, or zero, because of the denominator):

$$C' = 6x - \frac{48}{x^2} = \frac{6x^3 - 48}{x^2} = 6\frac{x^3 - 8}{x^2}.$$

This is undefined when $x = 0$, which is not in our domain. It is zero when $x^3 = 8$, which means $x = 2$. So $x = 2$ is the only critical point. Since $C' < 0$ when $x < 2$ and $C' > 0$ when $x > 2$, the cost function $C(x)$ has a local minimum at $x = 2$; since $x = 2$ is the unique critical point, S in fact has an absolute minimum there.

Finally (since $h = 4/x^2$), when $\boxed{x = 2,\ h = 1}$. These are the dimensions of the box.

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4. Use implicit differentiation to find the slope $\frac{dy}{dx}$ of the curve given by

$$x^3 - xy^2 - \cos y = 1$$

at the two points (π, π) and $(\pi, -\pi)$.

Solution. Implicitly differentiate:

$$3x^2 - y^2 - 2xy\frac{dy}{dx} + \frac{dy}{dx}\sin y = 0.$$

Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(-2xy + \sin y) = -3x^2 + y^2,$$

so

$$\frac{dy}{dx} = \frac{-3x^2 + y^2}{-2xy + \sin y}.$$

At the point (π, π), this is

$$\frac{-3\pi^2 + \pi^2}{-2\pi^2 + \sin\pi} = \frac{-2\pi^2}{-2\pi^2} = \boxed{1}.$$

At the point $(\pi, -\pi)$, this is

$$\frac{-3\pi^2 + \pi^2}{2\pi^2 + \sin(-\pi)} = \frac{-2\pi^2}{2\pi^2} = \boxed{-1}.$$

(Notice that these answers make sense when compared to the picture.)

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5. (a) $\lim_{x\to\infty} \left(\frac{\sin x}{\ln x}\right)$

Solution. The numerator oscillates between -1 and 1. The denominator goes to ∞. The quotient of a number between -1 and 1 and a very large number is very small: the limit is $\boxed{0}$.

(b) $\lim_{x\to 1} \frac{x^\pi - 1}{x^{\sqrt{2}} - 1}$.

Solution. As x goes to 1, the top and bottom both go to zero: this is an indeterminate form of the type $\frac{0}{0}$. So use L'Hôpital's rule:

$$\lim_{x\to 1} \frac{x^\pi - 1}{x^{\sqrt{2}} - 1} = \lim_{x\to 1} \frac{\pi x^{\pi-1}}{\sqrt{2}x^{\sqrt{2}-1}}.$$

This second quotient is defined and continuous when $x = 1$, so plug in $x = 1$: the answer is $\boxed{\frac{\pi}{\sqrt{2}}}$.

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6. Consider the function $f(x) = \begin{cases} 1 - \cos x & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$

(i) Is f continuous at $x = 0$?

(ii) Is f' defined at $x = 0$?

(iii) Is f'' defined at $x = 0$?

Solution. (i) $\boxed{\text{Yes}}$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0,$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1 - \cos x) = 0.$$

Since the right- and left-hand limits agree, $\lim_{x\to 0} f(x) = 0 = f(0)$: that is, the limit as x approaches 0 of $f(x)$ exists, and equals $f(0)$. That's the definition of continuity at $x = 0$.

(ii) $\boxed{\text{Yes}}$. I can differentiate the pieces of the function, to get this:

$$f'(x) = \begin{cases} \sin x & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

When $x = 0$, these two formulas agree (and are zero), so the derivative $f'(0)$ exists (and equals zero).

Alternatively, the derivative at $x = 0$ is defined by the formula

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}.$$

I'll investigate this limit by computing the right- and left-hand limits.

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{0 - 0}{x} = \lim_{x\to 0^-} 0 = 0.$$

Also,

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{(1 - \cos x) - 0}{x} = \lim_{x\to 0^+} \frac{1 - \cos x}{x}.$$

This last limit is an indeterminate form of type $\frac{0}{0}$, so compute it with L'Hôpital's rule: it is equal to

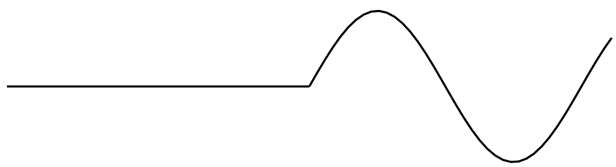
$$\lim_{x\to 0^+} \frac{\sin x}{1} = 0.$$

(Alternatively, recognize this as the limit computing the derivative of $1 - \cos x$ when $x = 0$.) So the one-sided limits agree; therefore the ordinary limit exists. Therefore the derivative exists.

(iii) $\boxed{\text{No}}$. From the calculations in part (ii), I have a formula for $f'(x)$:

$$f'(x) = \begin{cases} \sin x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

This function has a "corner" at $x = 0$; that is, it looks like this:



There is no tangent line at $x = 0$, so the derivative of this function does not exist.

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7. Let $f(x) = \sqrt{x+2}$, where $x \geq -2$. Find $f'(2)$ using limits and the definition of the derivative.

Solution.

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{(2+h)+2} - \sqrt{2+2}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \\ &= \lim_{h \to 0} \frac{4 + h - 4}{h(\sqrt{4+h} + 2)} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{4+h} + 2)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{4+h} + 2}. \end{aligned}$$

Now I can plug in $h = 0$, to get $\frac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$.

Note: I can check this using derivative formulas: $f'(x) = \frac{1}{2}(x+2)^{-1/2}$, so $f'(2) = \frac{1}{2}\frac{1}{\sqrt{4}} = \frac{1}{4}$. (I'm allowed to use derivative formulas to check my work, just not to solve the problem.)

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8. Calculamb: his height is given by

$$f(t) = e^{t-1}(1 - \cos 2\pi t), \quad t \geq 0.$$

(a) Find a point where his velocity is 0 meters per second.

Solution. This will happen at each minimum and each maximum. From the graph, it looks like this happens when $t = 0$, $t = 1$, $t = 2$, and $t = 3$, as well as at some point t a bit larger than 0.5, some t a bit larger than 1.5, and some t a bit larger than 2.5.

The formula for the velocity is

$$\begin{aligned} f'(t) &= e^{t-1}(1 - \cos 2\pi t) + e^{t-1}(2\pi \sin 2\pi t) \\ &= e^{t-1}(1 - \cos 2\pi t + 2\pi \sin 2\pi t). \end{aligned}$$

When $t = 0$, this is certainly 0 (since $\cos 0 = 1$ and $\sin 0 = 0$). So $\boxed{t = 0}$ is one answer.

(Other possible answers: $t = 1$, $t = 2$, and $t = 3$. It's hard to find the coordinates where the height hits a maximum.)

(b) What is Calculamb's acceleration at $t = 2$ seconds?

Solution. Acceleration is the derivative of velocity:

$$a(t) = f''(t) = e^{t-1}(1 - \cos 2\pi t + 4\pi \sin 2\pi t + 4\pi^2 \cos 2\pi t)$$

(after using the product rule and doing a little algebra). Now plug in $t = 2$:

$$f''(2) = e^{2-1}(1 - \cos 4\pi + 4\pi \sin 4\pi + 4\pi^2 \cos 4\pi) = e(1 - 1 + 0 + 4\pi^2).$$

So the answer is $\boxed{4\pi^2 e}$.

Notice that this is positive, and the graph is concave up when $t = 2$, so the answer is plausible.

(c) Is Calculamb's upward speed increasing, decreasing, or both, on the interval $[2, 2.25]$?

Solution. $f''(t) = e^{t-1}(1 + (4\pi^2 - 1)\cos 2\pi t + 4\pi \sin 2\pi t)$. When $2 \leq t \leq 2.25$, $2\pi t$ is between 2π and $2\pi + \frac{\pi}{2}$. For this range of values, $\cos 2\pi t$ and $\sin 2\pi t$ both range between 0 and 1. So $f''(t)$ is positive for all t in $[2, 2.25]$. Thus $f'(t)$, which is the upward velocity, is $\boxed{\text{increasing}}$ throughout the interval.

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9. A circular oil slick of uniform thickness contains 100cm^3 of oil.

(a) The volume of the oil remains constant. Use the equation for the volume of a cylinder to relate the thickness to the radius.

Solution. Let r be the radius, h the thickness. Then the volume is $V = \pi r^2 h$. The volume here is 100, so here is an equation relating the thickness to the radius:

$$\boxed{\pi r^2 h = 100}.$$

(Also correct: $h = \frac{100}{\pi r^2}$.)

(b) As the oil spreads the thickness is decreasing at the rate of 0.01 cm/min. At what rate is the radius of the slick increasing when it is 10cm?

Solution. Differentiate the equation from part (a) with respect to time, t:

$$2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt} = 0.$$

Solve for $\frac{dr}{dt}$: after a little algebra, you get

$$\frac{dr}{dt} = \frac{-r}{2h} \frac{dh}{dt}.$$

I know that $h = \frac{100}{\pi r^2}$, so when $r = 10$, $h = \frac{1}{\pi}$. I also know that $\frac{dh}{dt} = -0.01$. So

$$\frac{dr}{dt} = \frac{-10}{1/2\pi}(-0.01) = \boxed{0.05\pi}.$$