Math 124 Exam 2 answers

20 November 2001

1. (20 points) Compute these derivatives:

   (a) $\dfrac{d}{dx}(\tan(3x)) =$

   Solution. $\boxed{3\sec^2(3x)}$ (chain rule).

   (b) If $y = x^{\sqrt{2x}}$, then $y' =$

   Solution. Use logarithmic differentiation: take logs of both sides: $\ln y = \ln(x^{\sqrt{2x}})$, so $\ln y = \sqrt{2x}\ln(x)$. Now differentiate:

$$\begin{aligned}\frac{y'}{y} &= (\sqrt{2x})'\ln(x) + \sqrt{2x}(\ln(x))' \\ &= \frac{1}{\sqrt{2x}}\ln x + \frac{\sqrt{2x}}{x}.\end{aligned}$$

   Solve for y', replacing y by $x^{\sqrt{2x}}$:

$$\boxed{y' = x^{\sqrt{2x}}\left(\frac{1}{\sqrt{2x}}\ln x + \frac{\sqrt{2x}}{x}\right)}.$$

   (There are various equivalent ways of writing the answer.)

   (c) $\left(\dfrac{x^2+3}{e^{-x}-x}\right)' =$

   Solution¿ Quotient rule:

$$\boxed{\frac{(e^{-x}-x)(2x) - (x^2+3)(-e^{-x}-1)}{(e^{-x}-x)^2}}.$$

2. (20 points) Use implicit differentiation to find the x-coordinates of all points on the curve

$$2ye^y = x^4 - 2x^3 + x^2$$

   where the tangent line is horizontal.

   Solution. Differentiate both sides (left side by the product rule, right side by the power rule):

$$2y'e^y + 2yy'e^y = 4x^3 - 6x^2 + 2x.$$

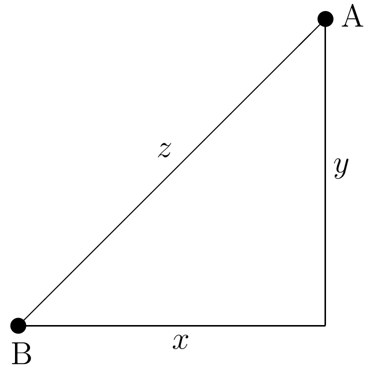
   Solve for y':

$$\begin{aligned}y' &= \frac{4x^3 - 6x^2 + 2x}{2e^y + 2ye^y} \\ &= \frac{2x^3 - 3x^2 + x}{e^y + ye^y}.\end{aligned}$$

The tangent line is horizontal when $y' = 0$. This quantity is zero when the numerator is zero—that is, when $2x^3 - 3x^2 + x = 0$. Factor out an x: $x(2x^2 - 3x + 1) = 0$. Factor the quadratic using the quadratic formula (or by eyeballing it): $x(2x - 1)(x - 1) = 0$. This is zero when $\boxed{x = 0}$, $\boxed{x = 1/2}$, or $\boxed{x = 1}$.

3. (20 points) At 3:00, two cars start moving from the same point; one ("car A") heads due north, the other ("car B") due west. Car A maintains a constant speed of 40 kilometers per hour. Car B's speed varies, though. At 3:45 (that is, 3/4 of an hour after they started), car B is stuck at a traffic light (not moving) 40 kilometers west of the starting point. How fast is the distance between the cars increasing at 3:45?

   Solution. Let $z(t)$ be the distance between the two cars at t hour after 3:00. We want to know $\frac{dz}{dt}$. Let x be the distance from B to its starting point; let y be the distance from A to its starting point. Here's a picture.

   So $z^2 = x^2 + y^2$. Differentiate with respect to t:

   $$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

   At 3:45, car A is 30 km north of its starting point: $y = 30$. Also, it is moving at 40 kph: $\frac{dy}{dt} = 40$. Car B is 40 km from its starting point: $x = 40$. It is stationary, so $dx/dt = 0$. Finally, by the Pythagorean theorem, $z = 50$. Plug all these numbers in:

   $$2 \cdot 50\frac{dz}{dt} = 2 \cdot 40 \cdot 0 + 2 \cdot 30 \cdot 40.$$

   So $\frac{dz}{dt} = 30 \cdot 40/50 = 24$. The answer is $\boxed{24}$ kph.

4. (20 points) A point moves along the y-axis in such a way that its position at time t is given by

   $$y(t) = 5te^{-t^2}.$$

   Assume $t \geq 0$.

   (a) Find all times $t \geq 0$ when the acceleration of the particle is zero.

   Solution. The acceleration is the second derivative of position, y''. The velocity is $y' = 5(e^{-t^2} - 2t^2e^{-t^2}) = 5e^{-t^2}(1 - 2t^2)$, so $y'' = 5(e^{-t^2}(-4t) - 2t(1 - 2t)e^{-t^2}) =$

$5e^{-t^2}(4t^3 - 6t)$. When is this zero? Since exponential functions are never zero, this is zero when $4t^3 - 6t = 0$. Factor this polynomial: $2t(2t^2 - 3) = 0$. The roots are $t = 0$, $t = -\sqrt{3/2}$, and $t = \sqrt{3/2}$. Since the problem asked for times $t \geq 0$, the answer is $\boxed{t = 0,\ t = \sqrt{3/2}}$.

(b) What is the velocity at those times?

Solution. I've already found the velocity: $y' = 5e^{-t^2}(1 - 2t^2)$. Plug in $t = 0$: $\boxed{y'(0) = 5}$. Plug in $t = \sqrt{3/2}$: $\boxed{y'(\sqrt{3/2}) = -10e^{-3/2}}$.

5. Two unrelated questions:

(a) (13 points) Compute the derivative of $\ln(\cos(\sqrt{x}))$.

Solution. Use the chain rule twice. The answer is:

$$\boxed{-\frac{1}{2}x^{-1/2}\sin(\sqrt{x})\frac{1}{\cos(\sqrt{x})}}.$$

(b) (7 points) Use implicit differentiation to derive the formula for the derivative of one of the following functions (your choice): $\ln x$, $\log_{10} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sin^{-1} x$. [Hint: let $y = f(x)$ where $f(x)$ is your chosen function, solve for x, and differentiate.]

Solution. I covered several of these (like $\ln x$, $\tan^{-1} x$, and $\sin^{-1} x$) in lecture, so you should have them in your notes. They are also covered in the book: see Sections 3.6 and 3.8. I'll derive the formula for $(\log_{10} x)'$; the others are similar.

Let $y = \log_{10} x$. Then $10^y = x$. Differentiate (implicitly) with respect to x: $y'10^y \ln 10 = 1$. So

$$y' = \frac{1}{10^y \ln 10} = \frac{1}{x \ln 10}.$$