Math 124 Exam 1 answers

23 October 2001

1. (20 points) Let $f(x) = \dfrac{1}{x}$. Find the equation of the line tangent to the curve $y = f(x)$ at the point $(2, \frac{1}{2})$. You must use limits to do this problem. Do not use formulas from high school calculus.

   Solution. To find the slope of the tangent line, I'll use the appropriate limit:

$$\lim_{h\to 0} \frac{f(2+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}.$$

   Put things over a common denominator:

$$= \lim_{h\to 0} \frac{\frac{2-(2+h)}{2(2+h)}}{h} = \lim_{h\to 0} \frac{-h}{2(2+h)h}.$$

   Now cancel the h's, at which point I can plug in $h = 0$ to compute the limit:

$$= \lim_{h\to 0} \frac{-1}{4+2h} = \frac{-1}{4}.$$

   That's the slope of tangent line. It goes through the point $(2, 1/2)$, so the equation is $y - 1/2 = (-1/4)(x-2)$, which (after a bit of algebra) is

$$\boxed{y = -\frac{1}{4}x + 1}.$$

2. (20 points) Determine if the following limits exist. If they exist, compute them. Justify your answers.

   (a) $\displaystyle\lim_{x\to\infty} \left(\frac{1}{x} - \frac{1}{x-1}\right)$

   Solution. As x goes to infinity, $1/x$ and $1/(x-1)$ both go to zero. So their difference goes to zero: the limit is $\boxed{0}$.

   (b) $\displaystyle\lim_{x\to 2} \frac{x^2 - 3x + 2}{x - 2}$

   Solution. The numerator and denominator both go to zero as x goes to 2, so I should look for some cancellation. This means: factor the numerator:

$$\lim_{x\to -1} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x\to -1} \frac{(x-1)(x-2)}{x-2} = \lim_{x\to -1} (x-1).$$

   At this point, I can plug in $x = 2$ to get the answer: $\boxed{1}$.

   (c) $\displaystyle\lim_{x\to \frac{\pi}{6}^-} \sin x$

   Solution. Since $\sin x$ is continuous at $x = \pi/6$, I can just plug it in to compute the limit: the answer is $\boxed{\sin(\pi/6) = 1/2}$.

(d) $\lim_{x \to 0} \csc x$

Solution. $\csc x$ is defined to be $1/(\sin x)$. As x goes to 0, $\sin x$ goes to zero, so $\lim_{x \to 0} \csc x$ does not exist. More precisely, as x goes to 0 from the left, $\csc x$ goes to $-\infty$; as x goes to 0 from the right, $\csc x$ goes to ∞. So the limit $\boxed{\text{does not exist}}$ (and it is not ∞).

3. (7 points) Let $f(x) = x^2 + 1$. Find the point $(a, f(a))$ on the curve $y = f(x)$ at which the tangent line is perpendicular to the line $y = 2x + 1$. You must use limits to do this problem. Do not use formulas from high school calculus.

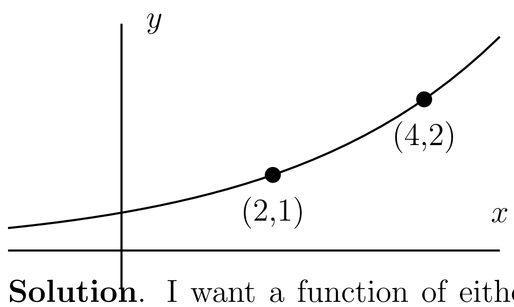
Solution. My approach is: find the slope of the tangent line at the point $(a, f(a))$; then I'll set it equal to $-1/2$, so that it's perpendicular to the given line of slope 2, and solve for a.

The slope of the tangent at $(a, f(a))$ is

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{((a+h)^2 + 1) - ((a)^2 + 1)}{h} \\
&= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 + 1 - a^2 - 1}{h} \\
&= \lim_{h \to 0} \frac{2ah + h^2}{h} \\
&= \lim_{h \to 0} \frac{2a + h}{1} \\
&= 2a.
\end{aligned}$$

I want this to equal $-1/2$, so a must be $-1/4$, in which case $f(a) = f(-1/4) = 17/16$. So the answer is: the point $\boxed{(-1/4, 17/16)}$.

4. (13 points) Here is a graph of an expontential function. Find a formula for this function.



Solution. I want a function of either the form $y = Ce^{kx}$ or the form $y = Ca^x$. The curve goes through the points $(2, 1)$ and $(4, 2)$. I'll do both cases.

$y = Ce^{kx}$: because of the two points, I get $1 = Ce^{2k}$ and $2 = Ce^{4k}$. Divide the second equation by the first: $2 = e^{2k}$. Take logs of both sides: $\ln 2 = 2k$. So $k = \ln 2$. Plug this back in to the first equation, $1 = Ce^{2 \ln 2}$, to find that $C = e^{-2 \ln 2}$. So the equation is

$$\boxed{y = e^{-2 \ln 2} e^{(\ln 2)x}}.$$

$y = Ca^x$: because of the two points, I get $1 = Ca^2$ and $2 = Ca^4$. Divide the second equation by the first: $2 = a^2$, so $a = \sqrt{2}$. Plug this back in to the first equation, $1 = C(\sqrt{2})^2$, to find that $C = 1/\sqrt{2}^2 = 1/2$. So the equation is

$$\boxed{y = \frac{1}{2}(\sqrt{2})^x}.$$

5. (20 points) Let $f(x) = \dfrac{x-1}{x+1}$.

   (a) What is the domain of $f(x)$?

   Solution. The only obstacle to $f(x)$ being defined is that it's a fraction, so its denominator might be zero. In fact, its denominator *is* zero when $x = -1$. At this point, $f(x)$ is not defined; its defined everywhere else. So the domain is $\boxed{\text{all points except } x = -1}$.

   (b) Find a formula for $f^{-1}(x)$.

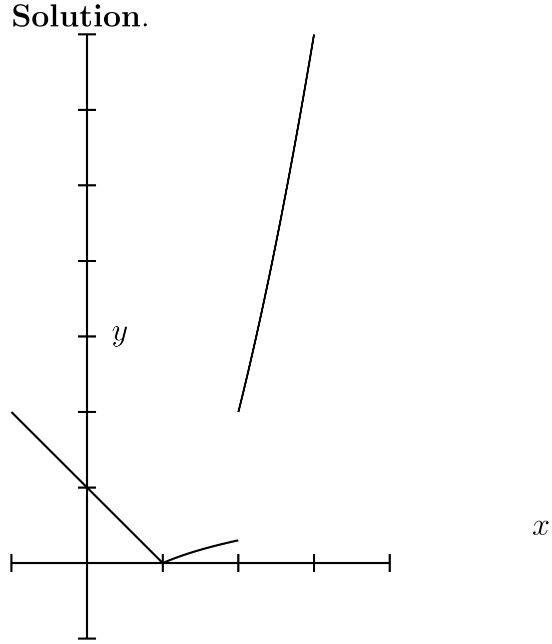
   Solution. Let $y = (x-1)/(x+1)$ and solve for x in terms of y: (x+1)y = x-1, so $xy + y = x - 1$, so $xy - x = -1 - y$, so $x = (-1-y)/(y-1)$. To express the answer, I'll switch the x's and y's (and cancel some minus signs):

$$\boxed{f^{-1}(x) = \frac{1+x}{1-x}}.$$

6. (20 points) Let $f(x) = \begin{cases} 1-x, & \text{if } x \le 1, \\ \ln x, & \text{if } 1 < x < 2, \\ x^2 - 2, & \text{if } 2 \le x. \end{cases}$

   (a) Sketch the graph of $f(x)$ on the domain $-1 \le x \le 3$.

   Solution.



(b) Where is $f(x)$ continuous? Explain.

Solution. $1 - x$ is continuous for all x, $\ln x$ is continuous whenever x is positive (which it is here, since I'm only look at $\ln x$ when $1 < x < 2$), and $x^2 - 2$ is continuous for all x. So the individual pieces of $f(x)$ are all continuous. So the only possible points of discontinuity are where the function switches from one definition to the next: the points $x = 1$ and $x = 2$. To determine continuity there, compute the right- and left-hand limits at those points; to compute these limits, plug these points into the function definition on each side of the points:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1 - x) = 1 - 1 = 0, \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \ln x = \ln(1) = 0.$$

So the function is continuous at $x = 1$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \ln x = \ln(2), \qquad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2 - 2) = 2^2 - 2 = 2.$$

Since $\ln(2) \neq 2$, the function is discontinuous at $x = 2$. So the answer is $\boxed{f(x) \text{ is continuous everywhere except } x = 2}$.