Math 124 Exam 1 answers

23 October 2001

1. (20 points) Let $f(x) = \dfrac{1}{x-1}$. Find the equation of the line tangent to the curve $y = f(x)$ at the point $(3, \frac{1}{2})$. You must use limits to do this problem. Do not use formulas from high school calculus.

   Solution. To find the slope of the tangent line, I'll use the appropriate limit:

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{\frac{1}{(3+h)-1} - \frac{1}{3-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}.$$

   Put things over a common denominator:

$$= \lim_{h\to 0} \frac{\frac{2-(2+h)}{2(2+h)}}{h} = \lim_{h\to 0} \frac{-h}{2(2+h)h}.$$

   Now cancel the h's, at which point I can plug in $h = 0$ to compute the limit:

$$= \lim_{h\to 0} \frac{-1}{4+2h} = \frac{-1}{4}.$$

   That's the slope of tangent line. It goes through the point $(3, 1/2)$, so the equation is $y - 1/2 = (-1/4)(x-3)$, which (after a bit of algebra) is

$$\boxed{y = -\frac{1}{4}x + \frac{5}{4}}.$$

2. (20 points) Determine if the following limits exist. If they exist, compute them. Justify your answers.

   (a) $\displaystyle\lim_{x\to\infty} \left(\frac{1}{x} - \frac{1}{x+2}\right)$

   Solution. As x goes to infinity, $1/x$ and $1/(x+2)$ both go to zero. So their difference goes to zero: the limit is $\boxed{0}$.

   (b) $\displaystyle\lim_{x\to -1} \frac{x^2+3x+2}{x+1}$

   Solution. The numerator and denominator both go to zero as x goes to -1, so I should look for some cancellation. This means: factor the numerator:

$$\lim_{x\to -1} \frac{x^2+3x+2}{x+1} = \lim_{x\to -1} \frac{(x+1)(x+2)}{x+1} = \lim_{x\to -1} (x+2).$$

   At this point, I can plug in $x = -1$ to get the answer: $\boxed{1}$.

   (c) $\displaystyle\lim_{x\to \frac{\pi}{4}^-} \tan x$

   Solution. Since $\tan x$ is continuous at $x = \pi/4$, I can just plug it in to compute the limit: the answer is $\boxed{\tan(\pi/4) = 1}$.

(d) $\lim_{x\to\frac{\pi}{2}} \sec x$

Solution. $\sec x$ is defined to be $1/(\cos x)$. As x goes to $\pi/2$, $\cos x$ goes to zero, so $\lim_{x\to\frac{\pi}{2}} \sec x$ does not exist. More precisely, as x goes to $\pi/2$ from the left, $\sec x$ goes to ∞; as x goes to $\pi/2$ from the right, $\sec x$ goes to $-\infty$. So the limit $\boxed{\text{does not exist}}$ (and it is not ∞).

3. (7 points) Let $f(x) = 2x^2 - 1$. Find the point $(a, f(a))$ on the curve $y = f(x)$ at which the tangent line is perpendicular to the line $y = \frac{1}{2}x + 1$. You must use limits to do this problem. Do not use formulas from high school calculus.

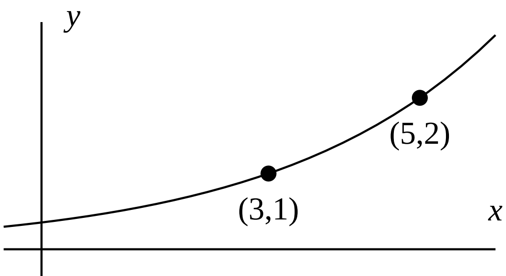
Solution. My approach is: find the slope of the tangent line at the point $(a, f(a))$; then I'll set it equal to -2, so that it's perpendicular to the given line of slope $1/2$, and solve for a.

The slope of the tangent at $(a, f(a))$ is

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{(2(a+h)^2 - 1) - (2(a)^2 - 1)}{h} \\
&= \lim_{h\to 0} \frac{2a^2 + 4ah + 2h^2 - 1 - 2a^2 + 1}{h} \\
&= \lim_{h\to 0} \frac{4ah + 2h^2}{h} \\
&= \lim_{h\to 0} \frac{4a + 2h}{1} \\
&= 4a.
\end{aligned}$$

I want this to equal -2, so a must be $-1/2$, in which case $f(a) = f(1/2) = -1/2$. So the answer is: the point $\boxed{(1/2, -1/2)}$.

4. (13 points) Here is a graph of an expontential function. Find a formula for this function.



Solution. I want a function of either the form $y = Ce^{kx}$ or the form $y = Ca^x$. The curve goes through the points $(3, 1)$ and $(5, 2)$. I'll do both cases.

$y = Ce^{kx}$: because of the two points, I get $1 = Ce^{3k}$ and $2 = Ce^{5k}$. Divide the second equation by the first: $2 = e^{2k}$. Take logs of both sides: $\ln 2 = 2k$. So $k = \ln 2$. Plug this back in to the first equation, $1 = Ce^{3\ln 2}$, to find that $C = e^{-3\ln 2}$. So the equation is

$$\boxed{y = e^{-3\ln 2} e^{(\ln 2)x}}.$$

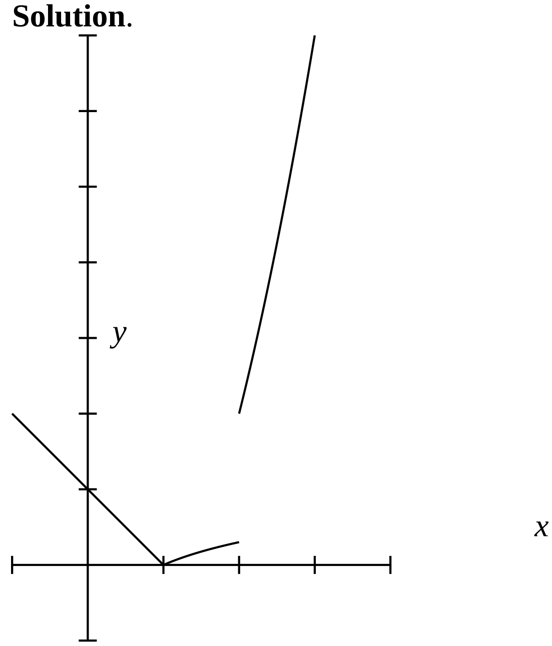
$y = Ca^x$: because of the two points, I get $1 = Ca^3$ and $2 = Ca^5$. Divide the second equation by the first: $2 = a^2$, so $a = \sqrt{2}$. Plug this back in to the first equation, $1 = C(\sqrt{2})^3$, to find that $C = 1/\sqrt{2}^3 = 1/\sqrt{8}$. So the equation is

$$\boxed{y = \frac{1}{\sqrt{8}}(\sqrt{2})^x}.$$

5. (20 points) Let $f(x) = \dfrac{x-2}{2x+1}$.
   (a) What is the domain of $f(x)$?
   Solution. The only obstacle to $f(x)$ being defined is that it's a fraction, so its denominator might be zero. In fact, its denominator *is* zero when $x = -1/2$. At this point, $f(x)$ is not defined; its defined everywhere else. So the domain is $\boxed{\text{all points except } x = -1/2}$.
   (b) Find a formula for $f^{-1}(x)$.
   Solution. Let $y = (x-2)/(2x+1)$ and solve for x in terms of y: (2x+1)y = x-2, so $2xy + y = x - 2$, so $2xy - x = -2 - y$, so $x = (-2-y)/(2y-1)$. To express the answer, I'll switch the x's and y's (and cancel some minus signs):

$$\boxed{f^{-1}(x) = \frac{2+x}{1-2x}}.$$

6. (20 points) Let $f(x) = \begin{cases} 1-x, & \text{if } x \le 1, \\ \ln x, & \text{if } 1 < x < 2, \\ x^2 - 2, & \text{if } 2 \le x. \end{cases}$
   (a) Sketch the graph of $f(x)$ on the domain $-1 \le x \le 3$.
   Solution.



(b) Where is $f(x)$ continuous? Explain.

Solution. $1-x$ is continuous for all x, $\ln x$ is continuous whenever x is positive (which it is here, since I'm only look at $\ln x$ when $1 < x < 2$), and $x^2 - 2$ is continuous for all x. So the individual pieces of $f(x)$ are all continuous. So the only possible points of discontinuity are where the function switches from one definition to the next: the points $x = 1$ and $x = 2$. To determine continuity there, compute the right- and left-hand limits at those points; to compute these limits, plug these points into the function definition on each side of the points:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1-x) = 1-1 = 0, \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \ln x = \ln(1) = 0.$$

So the function is continuous at $x = 1$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \ln x = \ln(2), \qquad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2-2) = 2^2 - 2 = 2.$$

Since $\ln(2) \neq 2$, the function is discontinuous at $x = 2$. So the answer is

$\boxed{f(x) \text{ is continuous everywhere except } x = 2}$.