Mathematics 124E, 124G

answer to bonus problem

Here is the equation of an ellipse: $x^2/9 + y^2/25 = 1$. Consider a point P lying on the ellipse in the first quadrant. Let L_N be the "normal line" through P. That is, L_N is the line through P which is perpendicular to the tangent line at P. Let x_N be the x-intercept of the normal line, and let y_N be its y-intercept. As P goes from $(3,0)$ to $(0,5)$, how do x_N and y_N behave?

First make a guess as to their behavior, just based on the picture and your intuition. Then use calculus to solve the problem. [It's probably easiest to use implicit differentiation.]

Solution. First, if I differentiate implicitly, I get this:

$$\frac{dy}{dx} = -\frac{25x}{9y}.$$

So if $P = (x_0, y_0)$ is a point on the ellipse, the slope of the tangent line at P is $-25x_0/9y_0$. This means that the slope of L_N is $9y_0/25x_0$, so I can write down the equation of L_N:

$$y - y_0 = \frac{9y_0}{25x_0}(x - x_0).$$

A little algebra yields this:

$$y = \frac{9y_0}{25x_0}x + \frac{16}{25}y_0.$$

The last term, $\frac{16}{25}y_0$, is the y-intercept y_N. As P goes from $(3,0)$ to $(0,5)$, y_0 goes from 0 to 5, so $\boxed{y_N \text{ goes from } 0 \text{ to } 16/5}$. (When $y_0 = 5$, the normal line is the y-axis, so there is no y-intercept. So y_N actually lies in the interval $[0, 16/5)$.)

To find the x-intercept x_N, set $y = 0$ and solve for x. The result is:

$$x_N = -\frac{16}{9}x_0.$$

Since x_0 is ranging from 3 to 0, $\boxed{x_N \text{ goes from } -16/3 \text{ to } 0}$. (Similarly, there is no x-intercept when $x_0 = 3$.)