Final Exam Information

Math 404, Spring 2000

The final exam is closed book, with no notes or calculators allowed. It will take place, as originally scheduled, 8:30–10:30 on Wednesday, June 7. The exam will consist of four problems taken from the following list, or possibly slight variations of these problems. I may also include a few other problems, either short answer (define some terms, determine whether some polynomials in $\mathbb{Q}[x]$ are irreducible, etc.) or essay-type questions (discuss Gauss' lemma and factorization in $\mathbb{Q}[x]$, discuss the prime elements in $\mathbb{Z}[i]$).

1. Let R be a ring, and prove the following: if $g(x)$ is a polynomial in $R[x]$, and if $\alpha \in R$ is a root of $g(x)$, then $x - \alpha$ divides $g(x)$ in $R[x]$.
2. Let p be a prime number congruent to 3 mod 4. Construct a field with p^2 elements.
3. Show that $\mathbb{Z}[x]$ is an integral domain, and determine its fraction field.
4. Let F be a field. True or false: $g(x) \in F[x]$ is irreducible if and only if it has no roots.
5. Partial fractions: see problems 11.1.10 and 11.1.13.
6. Prove that the power series ring $\mathbb{R}[\![t]\!]$ is a unique factorization domain.
7. Prove Proposition 11.2.11: If R is an integral domain, then every prime element of R is irreducible. If R is a principal ideal domain, then every irreducible element is prime.
8. Problem 11.Misc.1: prove that there are infinitely many primes congruent to 1 mod 4.
9. Problem 13.3.8: let F be a field, L an extension of F, α and β elements of L which are algebraic over F of degrees m and n, respectively. Let $K = F(\alpha, \beta)$. Show that if m and n are relatively prime, then $[K : F] = mn$.
10. Problem 13.4.2: prove in two ways that one can construct a regular pentagon with straightedge and compass.
11. Problem 13.4.4: is a regular 9-sided polygon constructible with straightedge and compass?

I think these two are harder. I may include them, but as optional problems or extra credit or something like that.

12. For which of these fields F are there irreducible polynomials in $F[x]$ of every positive degree: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$, $\ldots$, $\mathbb{F}_p$, $\ldots$?
13. Determine all positive integers which can be written as the sum of two squares (squares of integers, that is).