Mathematics 403 Exam solutions

1. Given the figure, identify the point group.

Let p be the center of one of the "empty" hexagons. Then you can rotate about p by the angle $2\pi/6$ and by any of its multiples. The figure has no reflections or glide reflections for symmetries, so the point group is isomorphic to

$$C_6 \cong \{1, \rho_{2\pi/6}, \rho_{4\pi/6}, \rho_{6\pi/6}, \rho_{2\pi/6}, \rho_{8\pi/6}, \rho_{10\pi/6}\}.$$

(Notice that since the translation group is a lattice, then the point group can't be C_5.)

2. What are the possible point groups when the translation group is a lattice?

C_n or D_n, with $n = 1$, 2, 3, 4, or 6.

3. Let G be a finite group of rotations of the plane about the origin. Show that G is cyclic.

Let θ be the smallest positive angle of rotation in G. I claim that every element of G is a power of ρ_θ; equivalently, I claim that if ρ_α is in G, then α is a multiple of θ. If α is not a multiple of θ, then α lies strictly between two adjacent multiples of θ: $k\theta < \alpha < (k+1)\theta$. Therefore, $0 < \alpha - k\theta < \theta$. Since ρ_α and ρ_θ are both in G, then so is

$$\rho_\alpha \rho_\theta^{-k} = \rho_{\alpha - k\theta}.$$

But this is a rotation in G by a smaller angle than θ. This is a contradiction, so the assumption that α is not a multiple of θ must be wrong.

4. Identify the group $\mathrm{Aut}(C_8)$.

Let $C_8 = \{1, x, x^2, \ldots, x^7\}$. Every homomorphism ϕ from C_8 to itself is determined by where x goes: if $\phi(x) = x^k$, then $\phi(x^2) = (x^k)^2 = x^{2k}$, and more generally, $\phi(x^i) = x^{ik}$. So there are eight different homomorphisms from C_8 to itself; such a homomorphism is an automorphism if it sends x to an element of order 8. There are four elements of order 8—x, x^3, x^5, and x^7—so there are four automorphisms:

- e, defined by $e(x^i) = x^i$.
- α, defined by $\alpha(x^i) = x^{3i}$.
- β, defined by $\beta(x^i) = x^{5i}$.
- γ, defined by $\gamma(x^i) = x^{7i}$.

Thus $\mathrm{Aut}(C_8) = \{e, \alpha, \beta, \gamma\}$. To finish the problem, I need to specify the group structure on this set. For example, to compute $\alpha^2 = \alpha \circ \alpha$, I look at

$$(\alpha \circ \alpha)(x) = \alpha(\alpha(x)) = \alpha(x^3) = x^9 = x.$$

Since $\alpha \circ \alpha$ sends x to itself, then $\alpha \circ \alpha = e$. Similarly, $(\beta \circ \beta)(x) = x^{25} = x$, so $\beta \circ \beta = e$; and $(\gamma \circ \gamma)(x) = x^{49} = x$, so $\gamma \circ \gamma = e$. Since this is a group of order 4 in which no element has order 4, it must be isomorphic to the Klein 4 group, $C_2 \times C_2$.

5. What are the orbits for the action of $O(2)$ on the plane $\mathbf{R}^2$?

If A is orthogonal, then multiplication by A is a rigid motion that fixes the origin: it is either a rotation about the origin or a reflection across a line through the origin. Hence (as we saw in Section 4.5), orthogonal matrices preserve length: the length of Av is the same as the length of v; thus every vector in the orbit of v has the same length as v. Furthermore, if v and w are two vectors with the same length, there is a rotation that carries v to w; hence they are in the same orbit.

So the orbit of any vector v is

$$O_v = \{w : |w| = |v|\}.$$

In other words, the orbits are the concentric circles around the origin.

6(a). What is the class equation of C_6?

Since $C_6 = \{1, x, x^2, x^3, x^4, x^5\}$ is abelian, every element is conjugate only to itself: $x^j x^i x^{-j} = x^i$. Thus the conjugacy class of x^i just contains x^i, so the decomposition of the group into conjugacy classes is

$$C_6 = \{1\} \cup \{x\} \cup \{x^2\} \cup \{x^3\} \cup \{x^4\} \cup \{x^5\},$$

and the class equation is

$$6 = 1 + 1 + 1 + 1 + 1 + 1.$$

6(b). What is the class equation of D_5?

Let's start with the conjugacy class of y; this is the set of all things of the form gyg^{-1}, where g is an element of D_5. Let $g = 1$; then $1y1^{-1} = y$, so y is conjugate to itself. (This is true in any group: the conjugacy class of any element a always contains a.) I could let $g = x$, then $g = x^2$, etc., but things will be fastest if I let $g = x^i$:

$$x^i y x^{-i} = x^i x^i y = x^{2i} y.$$

So I get these elements, as i goes from 1 to 4: $x^2 y$, $x^4 y$, $x^6 y = xy$, $x^{8y} = x^3 y$. So I know that the conjugacy class of y contains at least these elements: $\{y, xy, x^2 y, x^3 y, x^4 y\}$. I claim that it doesn't contain anything else. Here are two different reasons: you can either compute $(x^i y) y (x^i y)^{-1}$ and see that you get the same things, or you can observe that the centralizer of y is $Z(y) = \{1, y\}$; since the order of the centralizer times the order of the conjugacy class is the order of the whole group, then the conjugacy class contains exactly 5 elements.

On to the conjugacy class of x: $x^i x x^{-i} = x$, and $(x^i y) x (x^i y)^{-1} = x^i y x y x^{-i} = \ldots = x^{-1} = x^4$. So $C_x = \{x, x^4\}$.

Similarly, $C_{x^2} = \{x^2, x^3\}$.

As always, $C_1 = \{1\}$.

Thus

$$D_5 = \{1\} \cup \{x, x^4\} \cup \{x^2, x^3\} \cup \{y, xy, x^2 y, x^3 y, x^4 y\},$$

and the class equation is

$$10 = 1 + 2 + 2 + 5.$$