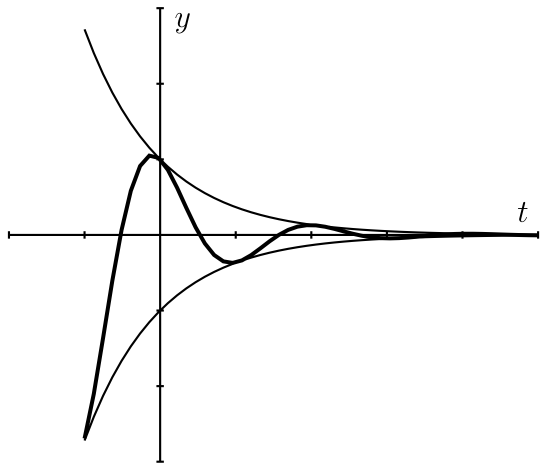
Exam 2 solutions

1. (a) (6 points) Draw a rough sketch of the function $y(t) = e^{-t}\cos(3t)$.

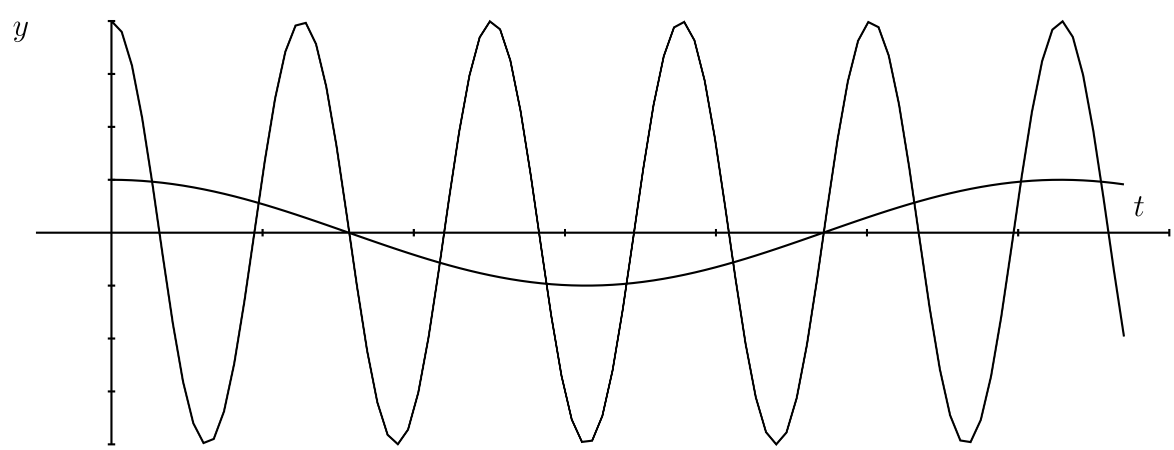
I was looking for a decaying exponential, with y-intercept 1 (i.e., when $t = 0$, $y = 1$). If you marked the axes, I was looking for a period of $2\pi/3$. I was also hoping for dashed lines indicating $y = e^{-t}$ and $y = -e^{-t}$, the two curves between which this one oscillates.



(b) (9 points) What are the differences between the graphs of the two functions

$$y_1(t) = \cos(t),$$
$$y_2(t) = 4\cos(5t - 1)?$$

Compared to y_1, y_2 has 4 times greater amplitude (so it oscillates between -4 and 4, instead of between -1 and 1), it has 5 times greater frequency (so 1/5 times the period, so it oscillates 5 times over the same time that y_1 oscillates once), and it has a phase angle of 1, which means that the graph is shifted to the right by the phase angle divided by the frequency: it's shifted to the right by 1/5. (The shift of 1/5 is too small to be visible on this picture, but anyway...)



2. (20 points) Find the general solutions to these differential equations:

   (a) $y'' - 5y' + \frac{25}{4}y = 0$.

   The characteristic equation if $r^2 - 5r + 25/4 = 0$, which has one root, $r = 5/2$. So the general solution is
$$\boxed{y = c_1 e^{5t/2} + c_2 t e^{5t/2}}.$$

   (b) $y'' - 2y' + 2y = 0$.

   The characteristic equation is $r^2 - 2r + 2 = 0$, which has roots $r = 1 \pm i$. So the general solution is
$$\boxed{y = c_1 e^t \cos t + c_2 e^t \sin t}.$$

3. (10 points) A certain driven mass-spring system is governed by this equation:
$$2u'' + \frac{1}{10}u' + ku = 5\cos 2t.$$

(So $u(t)$ is the position of the mass at time t.) The spring constant k is adjustable. What value for k, approximately, will maximize the amplitude of the oscillations of the mass? (The better your approximation, the more points it's worth.)

The answer is $\boxed{k = 8}$. In general, if the frequency ω of the driving force is allowed to vary, then in a mass-spring system with a small amount of damping, to maximize the amplitude, you want to set ω to be close to, but slightly less than $\omega_0 = \sqrt{k/m}$, the natural frequency of the system. In this case, ω is fixed, not varying—$\omega = 2$—while $\omega_0 = \sqrt{k/2}$ is varying. So the first guess might be that you should pick k so that ω_0 is a bit bigger than ω. Doing the algebra leads to the answer that k should be close to, but a bit bigger than, 8. A better approach is to use the formula on the last page of the exam for the particular solution to this sort of problem: the amplitude of that particular solution is
$$\frac{5}{\sqrt{4(k/2-4)^2 + \frac{1}{100}4}} = \frac{5}{\sqrt{(k-8)^2 + \frac{1}{25}}}.$$

To maximize this, I should minimize the denominator, and the denominator is minimized when $k = 8$.

4. (20 points) Here is a nonhomogeneous differential equation:
$$y'' - 3y' + 2y = f(t).$$

   (a) What is y_h, the solution to the associated homogeneous equation?

   $\boxed{y_h = c_1 e^t + c_2 e^{2t}}$.

For the remaining parts, I'll tell you $f(t)$, and I want you to tell me what to try for y_p, according to the method of undetermined coefficients. You don't have to solve for the coefficients, just tell me the right form.

(b) If $f(t) = 5e^t$, what should you try for y_p?

Since e^t is a solution to the homogeneous equation, I should try $\boxed{y_p = Ate^t}$.

(c) If $f(t) = 3e^{-t}$, what should you try for y_p?

e^{-t} is not a solution to the homogeneous equation, so $\boxed{y_p = Ae^{-t}}$ will work.

(d) If $f(t) = 2e^{2t} + 7\cos 6t - t^2$, what should you try for y_p?

e^{2t} is a solution to the homogeneous equation, so use

$$\boxed{y_p = Ate^{2t} + B\cos 6t + C\sin 6t + Dt^2 + Et + F}.$$

(e) If $f(t) = 2te^{3t}\sin 5t$, what should you try for y_p?

None of the pieces are solutions to the homogeneous equation, so use

$$\boxed{y_p = (At + B)e^{3t}(C\cos 5t + D\sin 5t)}.$$

5. (10 points) Consider the differential equation

$$t^2y'' + ty' - 4y = 0.$$

(a) Verify that $y = t^2$ is a solution.

Just plug $y = t^2$ in: $y' = 2t$, and $y'' = 2$, so the left side of the equation is $t^2(2) + t(2t) - 4(t^2)$. This is zero, the way it's supposed to be.

(b) Find the general solution.

Use reduction of order: let $y = vt^2$ and solve for the function v. $y' = v't^2 + 2vt$, and $y'' = v''t^2 + 4v't + 2v$, and when I plug this in to the equation, I get

$$t^2(v''t^2 + 4v't + 2v) + t(v't^2 + 2vt) - 4(vt) = 0.$$

The v terms cancel off, and I'm left with

$$t^4v'' + 5t^3v' = 0.$$

Let $w = v'$; then the equation is $t^4w' + 5t^3w = 0$, which is first order linear in w. (It is also separable; use whichever method you prefer for solving it.) The solution is $w = ct^{-5}$, so $v' = ct^{-5}$. Integrate to get v: $v = ct^{-4} + d$. Finally, $y = vt^2$, so $\boxed{y = ct^{-2} + dt^2}$.

6. (10 points) Find the general solution to $2y'' + 4y' + 10y = 2e^{-t}\sec 2t$.

Use variation of parameters. First divide by 2 so the equation is in its standard form: $y'' + 2y' + 5y = e^{-t}\sec 2t$. The associated homogeneous equation is $y'' + 2y' + 5y = 0$, which has solution $y_h = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$. Let $y_1 = e^{-t}\cos 2t$ and $y_2 = e^{-t}\sin 2t$. Then the Wronskian is

$$\begin{aligned} W = y_1y_2' - y_1'y_2 &= e^{-t}\cos 2t(2e^{-t}\cos 2t - e^{-t}\sin 2t) - (-2e^{-t}\sin 2t - e^{-t}\cos 2t)e^{-t}\sin 2t \\ &= 2e^{-2t}\cos^2 2t + 2e^{-2t}\sin^2 2t = 2e^{-2t}. \end{aligned}$$

So

$$
\begin{aligned}
y_p &= -e^{-t}\cos 2t \int \frac{e^{-t}\sin 2t e^{-t}\sec 2t}{2e^{-2t}}dt + e^{-t}\sin 2t \int \frac{e^{-t}\cos 2t e^{-t}\sec 2t}{2e^{-2t}}dt \\
&= -\frac{1}{2}e^{-t}\cos 2t \int \sin 2t \sec 2t\, dt + \frac{1}{2}e^{-t}\sin 2t \int \cos 2t \sec 2t\, dt \\
&= -\frac{1}{2}e^{-t}\cos 2t \int \tan 2t\, dt + \frac{1}{2}e^{-t}\sin 2t \int dt \\
&= -\frac{1}{4}e^{-t}\cos 2t \ln|\sec 2t| + \frac{1}{2}te^{-t}\sin 2t.
\end{aligned}
$$

Add this to y_h to get the general solution.

7. (10 points) Here is the equation of a mass-spring system: $3u'' + 2u' + 4u = 0$.

   (a) Is this underdamped, critically damped, overdamped?

   The roots of the characteristic equation are imaginary, so it's $\boxed{\text{underdamped}}$.

   (b) Suppose the initial conditions are $u(0) = 0$, $u'(0) = 2$. Draw a rough sketch of the function $u(t)$. (You should be able to do this without solving the differential equation.)

   I was looking for a decaying sine or cosine, starting at the origin, with initial slope 2.

8. (**Bonus**) Find the general solution for the differential equation $y''' - y = t^3$.

The characteristic equation is $r^3 - 1 = 0$, the roots of which are all numbers r so that $r^3 = 1$. You can solve this using the complex number stuff from earlier in the quarter, or you can factor out $r - 1$ and see that you get $r^3 - 1 = (r-1)(r^2 + r + 1) = 0$. This has roots $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. So

$$
y_h = c_1e^t + c_2e^{-t/2}\cos(\frac{\sqrt{3}}{2}t) + c_3e^{-t/2}\sin(\frac{\sqrt{3}}{2}t).
$$

You find y_p by trying a cubic: $y_p = At^3 + Bt^2 + Ct + D$. Plugging this in yields the formulas $A = -1$, $B = C = 0$, and $D = 6A$, so $D = -6$. So the answer is

$$
\boxed{y = c_1e^t + c_2e^{-t/2}\cos(\frac{\sqrt{3}}{2}t) + c_3e^{-t/2}\sin(\frac{\sqrt{3}}{2}t) - t^3 - 6}.
$$