Exam 1 solutions

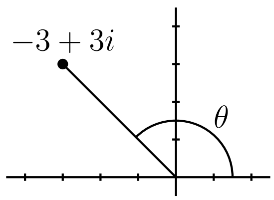
1. Write $\dfrac{1+i}{2-3i}$ in rectangular coordinates.

   Okay:

$$\frac{1+i}{2-3i} = \frac{1+i}{2-3i}\,\frac{2+3i}{2+3i} = \frac{(1+i)(2+3i)}{2^2+3^2} = \frac{-1+5i}{13} = \boxed{\frac{-1}{13} + i\frac{5}{13}}.$$

2. Plot the point $-3+3i$, and express it in polar coordinates.

   Here's the plot:



   By the Pythagorean Theorem, the "length" is $\sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2}$. The angle is $\theta = 3\pi/4$. Thus the answer is $-3+3i = \boxed{3\sqrt{2}e^{3\pi i/4}}$.

3. Find all complex numbers z that satisfy the equation $z^4 = -1$.

   Let $z = re^{i\theta}$, so that $z^4 = r^4e^{i4\theta}$. In polar coordinates, the equation $z^4 = -1$ becomes $r^4e^{i4\theta} = e^{i\pi}$. When two complex numbers are equal, their lengths are equal, so in this case, $r^4 = 1$. r must be real and positive, so $r = 1$. Also, their angles must be equal, or at least must differ by an integer multiple of 2π, so in this case, $4\theta - \pi$ is a multiple of 2π: $4\theta - \pi = \cdots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$. Solve for θ: $\theta = \cdots, -3\pi/4, -\pi/4, \pi/4, 3\pi/4, 5\pi/4, \cdots$. All but four of these are redundant; those four give us the four solutions: $\boxed{z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}}$.

4. Find the general solution to the differential equation $y'' - 2y' + 5y = 0$.

   The characteristic equation is $r^2 - 2r + 5 = 0$, which has roots $r = 1 \pm 2i$. So one way to write the general solution is $y = a_1e^{(1+2i)t} + a_2e^{(1-2i)t}$. A better way to write it is $\boxed{y = c_1e^t\cos 2t + c_2e^t\sin 2t}$.

5. Consider this initial value problem: $y'' - 6y' = 0$, $y(0) = -2$, $y'(0) = -18$.

   (a) Solve the initial value problem.

   The characteristic equation is $r^2 - 6r = 0$, which factors as $(r-6)r = 0$. This has roots $r = 6$ and $r = 0$, so the general solution is $y = c_1e^{6t} + c_2e^{0t} = c_1e^{6t} + c_2$. So $y' = 6c_1e^{6t}$. The initial conditions give the equations

$$\begin{aligned} -2 &= c_1 + c_2, \\ -18 &= 6c_1, \end{aligned}$$

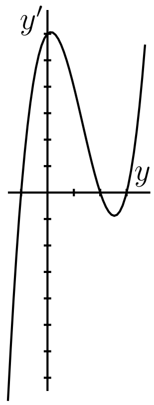
   so $c_1 = -3$ and $c_2 = 1$. So the solution is $\boxed{y = -3e^{6t} + 1}$.

(b) Verify that your answer is correct.

Given the formula for y, then $y' = -18e^{6t}$ and $y'' = -(18 \cdot 6)e^{6t}$. So $y'' - 6y' = -(18 \cdot 6)e^{6t} - 6(-18e^{6t})$ is in fact zero, so y is a solution. Also, $y(0) = -3 + 1 = -2$, and $y'(0) = -18$. So it is the right answer.

6. Here is a differential equation: $\dfrac{dy}{dt} = (y+1)(y-2)(y-3)$. **Do not solve it.** Instead, analyze it like some of the population problems in the homework. (In most of those problems, y was assumed to be positive. In this problem, there is no such restriction: y may be positive, negative or zero.)

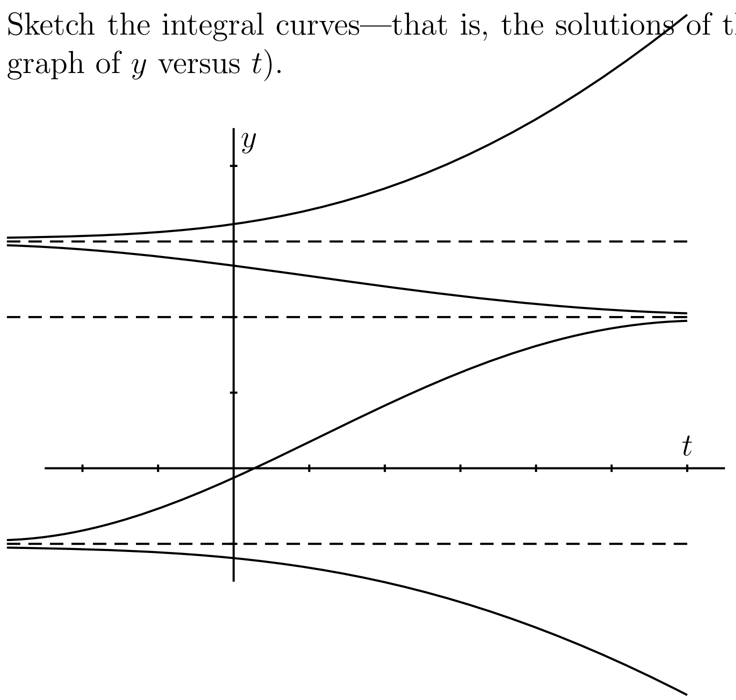
(a) Draw a rough sketch of the graph of dy/dt versus y.



(b) Determine the critical points, and classify each one as either stable or unstable.

The critical points are the points where $dy/dt = 0$. These are $\boxed{y=-1}$, $\boxed{y=2}$, and $\boxed{y=3}$. Since dy/dt is negative when $y < -1$, and since it is positive when $-1 < y < 2$, then $y = -1$ is an unstable critical point. Since dy/dt is negative when $2 < y < 3$, $y = 2$ is stable, and similarly, $y = 3$ is unstable.

(c) Sketch the integral curves—that is, the solutions of the differential equation (on a graph of y versus t).



7. Solve the initial value problem $y' = 2y^2 + xy^2$, $y(0) = 1$.

This is a separable equation. Rewrite it as $y' = y^2(2 + x)$, or

$$\frac{dy}{y^2} = (2 + x)dx.$$

Integrate both sides: $-1/y = x^2/2 + 2x + c$. Solve for y:

$$y = \frac{-1}{x^2/2 + 2x + c}.$$

The initial condition tells me that $1 = -1/c$, so $c = -1$, and the solution is

$$\boxed{y = \frac{-1}{x^2/2 + 2x - 1}}.$$

8. Solve the initial value problem $y' + y = 5\sin 2t$, $y(0) = 0$.

This is a first order linear equation. Since the coefficient of y is 1, the integrating factor is $e^{\int 1dt} = e^t$. Multiply by this: $e^t y' + e^t y = 5e^t \sin 2t$. Integrate both sides: the left side is the integrating factor times y, and the right side was given in the formulas on the last page. You should get

$$e^t y = 5(\frac{e^t}{5}(\sin 2t - 2\cos 2t) + c,$$

so

$$y = \sin 2t - 2\cos 2t + ce^{-t}.$$

The initial condition says that $0 = -2 + c$, so $c = 2$, and the solution is

$$\boxed{y = \sin 2t - 2\cos 2t + 2e^{-t}}.$$

9. Consider the differential equation $y'' + 2y' + y = 0$. In this case, the characteristic equation is $r^2 + 2r + 1 = 0$, which has only one root, $r = -1$. As a result, $y_1(t) = e^{-t}$ is a solution. (You don't have to check that.)

   (a) Verify that $y_2(t) = te^{-t}$ is also a solution.

   $y_2' = e^{-t} - te^{-t}$ (by the product rule), and $y_2'' = -2e^{-t} + te^{-t}$, so $y_2'' + 2y_2' + y_2 = (-2e^{-t} + te^{-t}) + 2(e^{-t} - te^{-t}) + (te^{-t})$, and everything cancels, so you get zero. So it's a solution.

   (b) Use the Wronskian to determine whether the solutions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ are linearly independent.

   The formula for the Wronskian is $W = y_1 y_2' - y_1' y_2$, which in this case is

   $$W = (e^{-t})(e^{-t} - te^{-t}) - (-e^{-t})(te^{-t}) = e^{-2t}.$$

   This is not zero, so the two solutions are $\boxed{\text{linearly independent}}$.

10. (**Bonus**) Use Euler's formula to show that $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

Solution. By Euler's formula, $\cos\alpha = \operatorname{Re}(e^{i\alpha})$ for any number α. Let $\alpha = 3\theta$; then

$$\cos 3\theta = \operatorname{Re}(e^{i3\theta}).$$

So I'll compute $e^{i3\theta}$ and then find its real part.

$$\begin{aligned} e^{i3\theta} &= (e^{i\theta})^3 && \text{by laws of exponents} \\ &= (\cos\theta + i\sin\theta)^3 && \text{by Euler's formula} \\ &= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3(\cos\theta)(i\sin\theta)^2 + (i\sin\theta)^3 \\ & && \text{by multiplying it out, or by the binomial theorem} \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta. \end{aligned}$$

Now, $i^2 = -1$, so $i^3 = -i$; when I plug these in, I get

$$e^{i3\theta} = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

The real part of this (which is what we're looking for) is

$$\cos 3\theta = \operatorname{Re}(e^{3i\theta}) = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

Finally, $\sin^2\theta = 1 - \cos^2\theta$, and when I plug this in and do the algebra, I get the advertised formula:

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

By the way, the problem didn't ask for this, but now I can also find $\sin 3\theta$, since it is the imaginary part of $e^{3i\theta}$:

$$\begin{aligned} \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta. \end{aligned}$$