Summary: first order differential equations

Types discussed in class.

1. *Separable equations.* These are equations which may be written in the form
$$y' = f(y)g(t).$$
To solve, you separate the variables:
$$\frac{1}{f(y)}dy = g(t)dt.$$
Then integrate, making sure to include one of the constants of integration:
$$\int \frac{1}{f(y)}dy = \int g(t)dt + c.$$

2. *Linear equations.* These are equations of this form:
$$y' + p(t)y = q(t).$$
To solve, make sure it's in exactly this form, and multiply by the "integrating factor" $e^{\int p(t)dt}$:
$$e^{\int p(t)dt}y' + e^{\int p(t)dt}p(t)y = e^{\int p(t)dt}q(t).$$
Let $I(t)$ denote the integrating factor: $I(t) = e^{\int p(t)dt}$. Then the left side of the equation is the derivative of $I(t)y$:
$$(I(t)y)' = I(t)q(t).$$
Now integrate:
$$I(t)y = \int I(t)q(t)dt + c,$$
so
$$y = \frac{1}{I(t)}\int I(t)q(t)dt + \frac{c}{I(t)}.$$
(I find it much easier to remember the procedure—multiply by $e^{\int p(t)dt}$—than to try to memorize this solution.)

Types not discussed in class. You won't need to know these for any homework or exams for this class, but they might come up in other courses.

1. *Homogeneous equations.* (See Section 2.9.) (**Warning**: the word "homogeneous" gets used in several different ways when studying differential equations.) A homogeneous equation is one of this form:

   $$y' = f(x, y),$$

   where the function $f(x, y)$ depends only on the ratio y/x—say $f(x, y) = (y/x)^2 + \sin(3\,y/x)$. In other words, it is an equation of the form

   $$y' = F(y/x)$$

   for some function F. Let $v = y/x$, so that $y = vx$. Then $y' = v'x + v$, and if you plug this in for y' and plug v in for y/x, you get

   $$xv' + v = F(v).$$

   This is separable:

   $$\frac{dv}{F(v) - v} = \frac{dx}{x}.$$

   So solve it for v, and then substitute back in for y: $v = y/x$.

2. *Bernoulli equations.* (See problems 37–41 in Section 2.2.) These are a lot like linear equations; they are equations of this form:

   $$y' + p(t)y = q(t)y^n,$$

   where n is any number except 0 or 1. (If $n = 0$, then $y^0 = 1$, so this is just a linear equation. If $n = 1$, then $y^1 = y$, so you can rewrite this as $y' + [p(t) - q(t)]y = 0$, which is a linear equation.)

   To solve it, make the substitution $v = y^{1-n}$, so that $v' = (1-n)y^{-n}y'$; in other words, $y^{-n}y' = \frac{1}{1-n}v'$. Multiply the original equation by y^{-n}:

   $$y^{-n}y' + p(t)y^{1-n} = q(t).$$

   Now make the substitution with v:

   $$\frac{1}{1-n}v' + p(t)v = q(t).$$

   Multiply everything by $1 - n$ and you have a linear equation, which you can solve to find v. Once you have v, then use the equation $y = v^{1/(1-n)}$ to find y.

Another type not discussed in class. This may not make sense unless you know what partial derivatives are.

1. *Exact equations.* I'll start with what I want the solution to look like, and then come up with the form for the differential equation. If you have an equation like this:
$$\psi(x, y) = c,$$
then taking the derivative with respect to x gives
$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0, \qquad \text{where} \qquad \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right).$$
So an exact equation is one that look like this:
$$M(x, y) + N(x, y)y' = 0,$$
where
$$\frac{\partial}{\partial y}M(x, y) = \frac{\partial}{\partial x}N(x, y).$$
To solve it, you want to use M and N to find ψ. So first you integrate $M(x, y)$ with respect to x, treating y as a constant: let
$$\psi(x, y) = \int M(x, y)dx + h(y)$$
for some function of y. (This $h(y)$ is the constant of integration, but since we were pretending that y was constant, it may depend on y.) It turns out that $h(y)$ must satisfy this equation:
$$h'(y) = N(x, y) - \int \frac{\partial}{\partial y}M(x, y)dx.$$
This may look complicated, but the right hand side always ends up depending only on y, so you just have to integrate it to get $h(y)$. Then $\psi(x, y) = c$ is an implicit solution of the original differential equation.

2. *Integrating factors, more generally.* Finally, suppose the equation looks like this
$$M(x, y) + N(x, y)y' = 0,$$
but where the condition on partial derivatives is not satisfied:
$$\frac{\partial}{\partial y}M(x, y) \neq \frac{\partial}{\partial x}N(x, y).$$
Sometimes it is possible to find a function $\mu(x, y)$, which is another sort of *integrating factor*, so that when you multiply by it:
$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)y' = 0,$$
the resulting equation is exact. There is no general procedure for finding $\mu(x, y)$, but there are methods that work some of the time. For example, let M_y be shorthand for $\frac{\partial}{\partial y}M$, and similarly for $N_x = \frac{\partial}{\partial x}N$. If
$$\frac{M_y - N_x}{N}$$
is a function only of x (no y involved), then you can find μ by solving this linear differential equation:
$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu.$$