## Math 126 Daily Fake Exam Problems Winter 2020

The deal: every day I'll post an exam-like problem at the start of class, along with the answer to the problem from the previous class. Please attempt these problems!

**DFEP #1: Wednesday, January 15th.**

Suppose $\mathbf{a} = \langle -1, 8, 4 \rangle$. Find a vector $\mathbf{b}$ so that:

- The angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$,
- $\mathbf{b}$ is perpendicular to $\mathbf{k}$, and
- $|\mathbf{b}| = 4$.

**DFEP #1 Solution:**

Let's say that $\mathbf{b} = \langle x, y, z \rangle$. We know that $\mathbf{b} \cdot \mathbf{k} = 0$, so $z = 0$.

We also know that $\mathbf{a} \cdot \mathbf{b} = -x + 8y + 4z = -x + 8y$. But on the other hand, $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cos(60°)$. Since $||\mathbf{a}|| = 9$ and $||\mathbf{b}|| = 4$, that means $-x + 8y = 18$, or $x = 8y - 18$.

Finally, since $||\mathbf{b}|| = 4$, we know that $x^2 + y^2 = 16$, so $(8y - 18)^2 + y^2 = 16$, which simplifies to $65y^2 - 288y + 308 = 0$.

Solving that tells us that $y = \dfrac{288 \pm \sqrt{288^2 - 4 \cdot 65 \cdot 308}}{130} \approx 2.627$ or $1.804$.

And since $x = 8y - 18$, that means we have two possible answers:

$$\mathbf{b} = \langle 3.016, 2.627, 0 \rangle \qquad \text{or} \qquad \mathbf{b} = \langle -3.570, 1.804, 0 \rangle$$

**DFEP #2: Friday, January 17th.**

(a) Give the equation of a plane containing the line $\dfrac{x-2}{4} = \dfrac{y}{-2} = \dfrac{z+6}{3}$ and the point $(6, 1, 5)$.

(b) Find the intersection of this plane with the line $\dfrac{x+1}{-6} = \dfrac{y-5}{2} = z - 7$.

**DFEP #2 Solution:**

(a) We want a plane through the line $\frac{x-2}{4} = \frac{y}{-2} = \frac{z+6}{3}$ and the point $(6, 1, 5)$. Certainly this plane contains the line's direction vector $\langle 4, -2, 3\rangle$. It also contains the points $(2, 0, -6)$ and $(6, 1, 5)$, which means it contains the vector $\langle 4, 1, 11\rangle$.

So to find the normal vector, we can take the cross product $\langle 4, -2, 3\rangle \times \langle 4, 1, 11\rangle$ to get $\langle -25, -32, 12\rangle$. The plane with normal vector $\langle -25, -32, 12\rangle$ through the point $(6, 1, 5)$ has equation

$$-25x - 32y + 12z = -25(6) - 32(1) + 12(5)$$

or

$$-25x - 32y + 12z = -122$$

(b) Let's write that line in parametric form: $x = -1 - 6t$, $y = 5 + 2t$, $z = 7 + t$. Plugging that into the equation of the plane yields

$$-25(-1 - 6t) - 32(5 + 2t) + 12(7 + t) = -122$$

which we can solve to get $t = -71/98 \approx -0.7245$, so the point of intersection is $(x, y, z) = (3.347, 3.551, 6.276)$.

**DFEP #3: Wednesday, January 22nd:**

Find the equation of an ellipsoid centered at $(0, 1, -2)$ that passes through the points $(8, 4, -2)$, $(0, -4, -2)$, and $(2, 2, 3)$.

**DFEP #3 Solution:**

We want an ellipsoid centered at $(0, 1, -2)$, so it should have the form

$$\frac{x^2}{a^2} + \frac{(y-1)^2}{b^2} + \frac{(z+2)^2}{c^2} = 1.$$

Plugging in $(0, -4, -2)$ we can see that $b = 5$, and then plugging in $(8, 4, -2)$ gives $a = 10$. Finally, plugging in $(2, 2, 3)$ we get

$$\frac{2^2}{10^2} + \frac{1^2}{5^2} + \frac{5^2}{c^2} = 1$$

So $\frac{25}{c^2} = \frac{23}{25}$, so $c^2 = \frac{625}{23}$ and we get the ellipsoid

$$\frac{x^2}{100} + \frac{(y-1)^2}{25} + \frac{23(z+2)^2}{625} = 1.$$

**DFEP #4: Friday, January 24th:**

Consider the vector function $\mathbf{r} = \langle t+1, 2^t, 3t+2t^2 \rangle$.

(a) Does the curve defined by $\mathbf{r}$ intersect the following line? If so, where?

$$\frac{x-15}{2} = y - 10 = 8 - z$$

(b) Suppose $\mathbf{r}$ intersects the surface $5x^2 + Cy^2 + 2z^2 = 1$ in the $yz$-plane. Solve for the constant $C$.

(c) Describe the surface from part (b). Your answer should be a short phrase.

**DFEP #4 Solution:**

(a) We want to find the intersection of the vector functions $\langle t+1, 2^t, 3t+2t^2\rangle$ and $\langle 15+2s, 10+s, 8-s\rangle$. So we set their components equal:

$$t+1 = 15+2s \qquad 2^t = 10+s \qquad 3t+2t^2 = 8-s$$

Yikes, let's ignore that second equation for now. Solving the first and third gives a quadratic $4t^2+7t-30=0$, which factors as $(4t+15)(t-2)=0$. So we have either $t=2, s=-6$ or $t=-15/4, s=-71/8$. Plugging those into the second equation, we have $t=2, s=-6$ as the only solution.

So where's the point? Plug $t$ or $s$ into the corresponding vector function to get $(3,4,14)$ as the intersection.

(b) Okay, $\mathbf{r} = \langle t+1, 2^t, 3t+2t^2\rangle$ intersects the $yz$-plane when $x=0$, so $t=-1$, which is at the point $(0, \frac{1}{2}, -1)$. Since this intersects the curve $5x^2+Cy^2+2z^2=1$, we have $C\left(\frac{1}{2}\right)^2+2=1$, so $C=-4$.

(c) The curve $5x^2-4y^2+2z^2=1$ is a hyperboloid of one sheet, centered around the $y$-axis.

**DFEP #5: Monday, January 27th.**

Consider the curve defined by the vector function $\mathbf{r} = \langle t+6, t^3, e^{t^2-6t+8}\rangle$.

(a) Find all points where the curve intersects the plane $z=1$.

(b) Find the (acute) angle between the curve and the normal vector to the plane at each point from part (a).

**DFEP #5 Solution:**

(a) The curve defined by $\langle t+6, t^3, e^{t^2-6t+8}\rangle$ intersects $z = 1$ when its $z$-component is 1, which means that $e^{t^2-6t+8} = 1$. Therefore $t^2 - 6t + 8 = 0$, so $t = 2$ or $t = 4$.

To find the points of intersection, we plug $t = 2$ and $t = 4$ back into the vector function to get $(8, 8, 1)$ and $(10, 64, 1)$.

(b) We'll need to know the tangent vectors for the points from part (a). The derivative $\mathbf{r}'(t) = \langle 1, 3t^2, (2t-6)e^{t^2-6t+8}\rangle$.

At $t = 2$, this is the vector $\langle 1, 12, -2\rangle$, and at $t = 4$ it's $\langle 1, 48, 2\rangle$.

To find the angle between the normal vector and the tangent vector:
$\langle 1, 12, -2\rangle \cdot \langle 0, 0, 1\rangle = ||\langle 1, 12, -2\rangle|| \cdot 1\cos(\theta)$, so $\theta = \cos^{-1}(-2/\sqrt{149}) \approx 99.43°$.
We probably want the acute angle, so we'll go with $80.57°$.

A similar calculation for the other point gives $87.61°$.

**DFEP #6: Wednesday, January 29th.**

Find all intersections of the polar curve $r = \cos^2(\theta)$ with the line $x = \dfrac{1}{4}$.

**DFEP #6 Solution:**

Okay, so we have the curve $r = \cos^2(\theta)$, and we want to know where $x = \frac{1}{4}$.

But $x = r\cos(\theta)$, so $r\cos(\theta) = \frac{1}{4}$, which means $\cos^3(\theta) = \frac{1}{4}$.

That means $\theta = \cos^{-1}\left(\frac{1}{\sqrt[3]{4}}\right)$ is one solution. Since $\cos(\theta) = \cos(-\theta)$, we know $-\cos^{-1}\left(\frac{1}{\sqrt[3]{4}}\right)$ is another solution. In both cases, $x = \frac{1}{4}$, and $y = r\sin(\theta) = \pm\cos^2\left(\cos^{-1}\left(\frac{1}{\sqrt[3]{4}}\right)\right)\sin\left(\cos^{-1}\left(\frac{1}{\sqrt[3]{4}}\right)\right)$, which simplifies (using a comparison triangle) to $\pm\frac{\sqrt{4^{2/3}-1}}{4}$. The two points, then are

$$\left(\frac{1}{4}, \frac{\sqrt{4^{2/3}-1}}{4}\right) \quad \text{and} \quad \left(\frac{1}{4}, -\frac{\sqrt{4^{2/3}-1}}{4}\right)$$

.

**DFEP #7: Friday, January 31st.**

Let $\mathbf{r}(t) = \left\langle 2^t - t, t^2 - 4t, \frac{1}{1+t^2}\right\rangle$. Find $\mathbf{T}(t)$ at the point $\left(27, 5, \frac{1}{26}\right)$.

**DFEP #7 Solution:**

We want $\mathbf{T}(t)$ when $\mathbf{r}(t) = \left\langle 2^t - t, t^2 - 4t, \dfrac{1}{1+t^2} \right\rangle$ at $\left(27, 5, \dfrac{1}{26}\right)$, which is at $t = 5$.

Now $\mathbf{r}'(t) = \left\langle \ln(2)2^t - 1, 2t - 4, \dfrac{-2t}{(1+t^2)^2} \right\rangle$, which at $t = 5$ is:

$$\mathbf{r}(t) = \left\langle 32\ln(2) - 1, 6, \frac{-5}{338} \right\rangle$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{32^2\ln(2)^2 - 64\ln(2) + 1 + 36 + \frac{5^2}{338^2}}} \left\langle 32\ln(2) - 1, 6, \frac{-5}{338} \right\rangle$$

**DFEP #8: Monday, February 3rd.**

The position of a bee over time on the interval $[0, \infty)$ is given by the vector function $\mathbf{r}(t) = \langle \cos(\pi t), t^4 - 4t^3 + 4t^2, \sqrt{t} \rangle$. Compute the tangential and normal acceleration of the bee after $t = 4$ seconds.

**DFEP #8 Solution:**

We are given the position vector $\mathbf{r} = \langle \cos(\pi t), t^4 - 4t^3 + 4t^2, \sqrt{t} \rangle$ and we want tangential and normal acceleration after $t = 4$ seconds.

First, we need $\mathbf{r}'(t) = \langle -\pi \sin(\pi t), 4t^3 - 12t^2 + 8t, 1/(2\sqrt{t}) \rangle$ (so $\mathbf{r}'(4) = \langle 0, 96, 1/4 \rangle$) as well as $\mathbf{r}''(t) = \langle -\pi^2 \cos(\pi t), 12t^2 - 24t + 8, -1/(4\sqrt{t^3}) \rangle$ (so $\mathbf{r}''(4) = \langle -\pi^2, 104, -1/32 \rangle$).

The usual formulas tell us $a_T$ and $a_N$:

$$a_T = \frac{r'(4) \cdot r''(4)}{|r'(4)|} = \frac{9983.99219}{\sqrt{96^2 + (1/4)^2}} \approx 103.9996$$

and

$$a_N = \frac{|r'(4) \times r''(4)|}{|r'(4)|} = \frac{|\langle -29, -\pi^2/4, 96\pi^2 \rangle|}{\sqrt{96^2 + (1/4)^2}} \approx 9.8742$$

**DFEP #9: Monday, February 10th.**

Compute the all the partial derivatives (one for each variable) of the given functions:

(a) $f(x, y) = x^2y^3 - xy + 5x^3$

(b) $g(x, y) = \dfrac{x^2 + 1}{xy + y^2}$

(c) $h(x, y, z) = (2 + \arctan(x + y^2))^z$

**DFEP #9 Solution:**

I don't really have anything to say about this one. Here are some derivatives.

(a) $f_x(x,y) = 2xy^3 - y + 13x^2$
$f_y(x,y) = 3x^2y^2 - x$

(b) $$g_x(x,y) = \frac{2x(xy+y^2) - y(x^2+1)}{(xy+y^2)^2}$$

$$g_y(x,y) = \frac{-(x^2+1)(x+2y)}{(xy+y^2)^2}$$

(c) $$h_x(x,y,z) = \frac{z(2+\arctan(x+y^2))^{z-1}}{1+(x+y^2)^2}$$

$$h_y(x,y,z) = \frac{2yz(2+\arctan(x+y^2))^{z-1}}{1+(x+y^2)^2}$$

$$h_z(x,y,z) = (2+\arctan(x+y^2))^z \ln(2+\arctan(x+y^2))$$

**DFEP #10: Wednesday, February 12th.**

Consider the surface $z = x^3e^y - 8\cos(y) + 4x\sin(y)$.

Let $P$ be the point where this surface intersects the $x$-axis.

Find the equation for the plane tangent to the surface at the point $P$.

**DFEP #10 Solution:**

We want the tangent plane to $z = x^3 e^y - 8\cos(y) + 4x\sin(y)$ at the point where it intersects the $x$-axis.

At that point, the $y$- and $z$-coordinates are zero, so we have $0 = x^3 - 8$, so $x = 2$. So the point is $(2, 0, 0)$.

What's the normal vector? We need the partial derivatives:

$$\frac{\partial z}{\partial x} = 3x^2 e^y + 4\sin(y) = 12$$

$$\frac{\partial z}{\partial y} = x^3 e^y + 8\sin(y) + 4x\cos(y) = 16$$

So we get the plane $z = 12(x - 2) + 16y$.

**DFEP #11: Friday, February 14th.**

Find all critical points of the function $f(x, y) = x + 3y - e^x - y^3$, and classify them as local minima, local maxima, or saddle points.

**DFEP #11 Solution:**

We need the critical points of $f(x, y) = x + 3y - e^x - y^3$, so we want to solve the equations:

$$f_x(x, y) = 1 - e^x = 0$$

$$f_y(x, y) = 3 - 3y^2 = 0$$

Which has two solutions: $(0, 1)$ and $(0, -1)$. Let's check $D(x, y)$ at each point:

The second derivatives are $f_{xx}(x, y) = -e^x$, $f_{yy}(x, y) = -6y$, and $f_{xy}(x, y) = 0$.

So $D(0, 1) = 6$ and $D(0, -1) = -6$. Since $f_{xx}(x, y) < 0$ for all $(x, y)$, that means $(0, 1)$ is a local maximum and $(0, -1)$ is a saddlepoint.

**DFEP #12: Wednesday, February 19th.**

Compute the volume of the solid between the plane $z = 0$ and the surface

$$z = y \sin(2y) \cos(xy)$$

over the region $[0, 2] \times [0, \pi/4]$.

**DFEP #12 Solution:**

We want to compute

$$\int_0^2 \int_0^{\pi/4} y\sin(2y)\cos(xy)\,dy\,dx$$

Oh, wait, that seems maybe impossible. Let's flip it around:

$$\int_0^{\pi/4} \int_0^2 y\sin(2y)\cos(xy)\,dx\,dy$$

That's easier: $y\sin(2y)$ is a constant, and the antiderivative of $\cos(xy)$ with respect to $x$ is $\sin(xy)/y$. So we get:

$$\int_0^{\pi/4} \left( \sin(2y)\sin(xy)\Big]_0^2 \right)\,dy$$

That's just

$$\int_0^{\pi/4} \sin^2(2y)\,dy = \int_0^{\pi/4} \frac{1}{2}(1-\cos(4y))\,dy$$

which comes out to $\pi/8$.

**DFEP #13: Friday, February 21st.**

Compute the double integral

$$\int_0^{e^9} \int_{\sqrt{\ln(y)}}^3 2xye^{x^2}\,dx\,dy$$

twice: once normally, and again by reversing the order of integration.

**DFEP #13 Solution:**

Oof, why did I give you this problem? First of all, it's straightforward enough as is:

$$\int_0^{e^9}\int_{\sqrt{\ln(y)}}^{3} 2xye^{x^2}\,dx\,dy = \int_0^{e^9}\left(ye^{x^2}\right)\Big]_{\sqrt{\ln(y)}}^{3} dy = \int_0^{e^9}(ye^9 - y^2)\,dy$$

which we can evaluate as

$$\frac{e^9}{2}y^2 - \frac{1}{3}y^3\Big]_0^{e^9} = \frac{e^{27}}{6}.$$

Did you notice that this is an improper integral, though? It totally is: $x = \sqrt{\ln(0)}$ is undefined. This would not happen on a midterm, probably.

Reversing the order of integration gives a much-more-obviously improper integral:

$$\int_{-\infty}^{3}\int_0^{e^{x^2}} 2xye^{x^2}\,dy\,dx$$

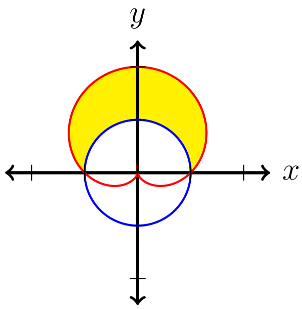
which also comes out to $\dfrac{e^{27}}{6}$.

**DFEP #14: Monday, February 24th.**

Compute the area inside the cardioid $r = \sin(\theta) + 1$ but outside the circle $x^2 + y^2 = 1$.

**DFEP #14 Solution:**

Here's a picture:




We want to integrate 1 over this domain. $\theta$ runs from 0 to $\pi$, and for any given $\theta$, $r$ runs from 1 to $1 + \sin(\theta)$. So we want:

$$\int_0^\pi \int_1^{1+\sin(\theta)} r\, dr\, d\theta = \int_0^\pi \left(\frac{1}{2}r^2\right)\Bigg]_1^{1+\sin(\theta)} d\theta = \frac{1}{2}\int_0^\pi (\sin^2(\theta) + 2\sin(\theta))\, d\theta$$

This becomes

$$\frac{1}{2}\int_0^\pi \left(\frac{1}{2}(1 - \cos(2\theta)) + 2\sin(\theta)\right) d\theta = \frac{1}{2}\left(\frac{x}{2} - \frac{1}{4}\sin(2\theta) - 2\cos(\theta)\right)\Bigg]_0^\pi$$

which simplifies to $\dfrac{\pi}{4} + 2$.