Axioms for the Real Numbers

This document will be updated throughout the quarter as we discuss more properties of $\mathbb{R}$, the set of real numbers.

In this document we will be listing **axioms**, basic facts about real numbers that we assume to be true. It might seem strange that we need to assume anything (after all, shouldn't we be able to prove this stuff?), but in mathematics we always need some set of facts to begin with; otherwise it's impossible to even begin a proof.

Field Axioms

The first group of axioms are due to the fact that the real numbers are a **field**, which is a term you can learn more about in Math 402. For now, you should know that a field is a set of objects in which we can perform **addition** (denoted by $a+b$) and **multiplication** (denoted by ab or $a \cdot b$) with the following properties:

Axiom 1. (Closure) If a and b are real numbers, then so is $a+b$ and ab.

Axiom 2. (Commutativity) For all real numbers a and b, $a+b=b+a$ and $ab=ba$.

Axiom 3. (Associativity) For all real numbers a, b, and c, $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.

Axiom 4. (Distributivity) For all real numbers a, b, and c, $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$.

Axiom 5. (Identities) There are two real numbers called 0 and 1 such that for every real number a, $a+0=0+a=a$ and $a\cdot 1=1\cdot a=a$.

Axiom 6. (Additive inverses) For every real number a, there exists an additive inverse called $-a$ such that $a+(-a)=-a+a=0$.

Axiom 7. (Multiplicative inverses) For every nonzero real number a, there exists a multiplicative inverse called a^{-1} such that $a\cdot a^{-1}=a^{-1}\cdot a=1$.

Note that the real numbers aren't the only field! The rational numbers also satisfy these axioms. So we definitely aren't done learning all of our axioms yet, since there are some other properties out there which distinguish the real numbers from the rational numbers.

Equality Axioms

We'll also use the following facts about **equality**. These ones don't just work for real numbers; they will be true any time we say that two things are "equal".

Axiom 8. (Reflexivity) For every real number a, $a=a$.

Axiom 9. (Symmetry) For all real numbers a and b, if $a=b$, then $b=a$.

Axiom 10. (Transitivity) For all real numbers a, b, and c, if $a=b$ and $b=c$ then $a=c$.

Axiom 11. (Substitution) For all real numbers a and b, if $a=b$, then b may be substituted for a in any number of places within a mathematical statement without changing the truth value of that statement.

Order Axioms

If we want to talk about some real numbers as being larger or smaller than others, we'll need the following two axioms, which define what it means for a real number to be positive:

Axiom 12. (Positive Closure) If a and b are positive, then so are $a+b$ and ab.

Axiom 13. (Trichotomy) If a is a real number, then exactly one of the following is true:

- a is positive.
- $-a$ is positive.
- $a = 0$.

Using these definitions, we can make the following definitions:

- $a < b$ means that $b - a$ is positive.
- $a \leq b$ means that $a < b$ or $a = b$.
- $a > b$ means that $a - b$ is positive.
- $a \geq b$ means that $a > b$ or $a = b$.
- a is "negative" that $-a$ is positive.
- The "maximum" of a set A is an element $a \in A$ such that for all $b \in A$, $a \geq b$.
- The "minimum" of a set A is an element $a \in A$ such that for all $b \in A$, $a \leq b$.

Integer Axioms

We're not done introducing axioms for the real numbers, but for now we'll pause and ask what axioms the integers have. The integers are a set $\mathbb{Z}$ with all of the same properties we've listed above, **except** for Axiom 7. In addition, the following three axioms are true:

Axiom 14. 1 is an integer.

Axiom 15. (Closure of $\mathbb{Z}$) If a and b are integers, then so are $a+b$, $a-b$, and ab.

Axiom 16. (Well-Ordering) Every nonempty set of positive integers has a minimum.

List of Theorems

Here are the theorems we've proven in class and on the homework.

Theorem 1. For all real numbers a, b, and c, if $a = b$, then $a + c = b + c$, and $ac = bc$.

Theorem 2. (Cancellation Law) For all real numbers a, b, and c, if $a + c = b + c$, then $a = b$.

Theorem 3. $-0 = 0$. That is, 0 is the additive inverse of 0.

Theorem 4. The additive identity is unique.

Theorem 5. Every real number has a *unique* additive inverse.

Theorem 6. $(-1)(-1) = 1$.

HW Theorem 1. For every real number a, $0a = 0$.

HW Theorem 2. For every real number a, $-a = (-1)a$.

HW Theorem 3. For all real numbers a and b, $(-a)(-b) = ab$.

Theorem 7. For every real number a, a is positive if and only if $a > 0$.

Theorem 8. (Trichotomy for Inequalities) For all real numbers a and b, exactly one of the following is true:

- $a < b$.
- $a > b$.
- $a = b$.

Theorem 9. (Transitivity of Inequality) Suppose a, b, and c are real numbers.

- If $a < b$ and $b < c$, then $a < c$.
- If $a < b$ and $b \leq c$, then $a < c$.
- If $a \leq b$ and $b < c$, then $a < c$.
- If $a \leq b$ and $b \leq c$, then $a \leq c$.

Theorem 10. 1 is positive.

Theorem 11. For all real numbers a, b, and c, if $a > b$ and c is positive, then $ac > bc$.

Theorem 12. For every real number a, if $0 < a < 1$, then $0 < a^2 < a$.

Theorem 13. 0 is an integer.

Theorem 14. 1 is the smallest positive integer.

Theorem 15. For every integer a, $a|a$.

Theorem 16. For all integers a, b, and c, if $a|b$ and $b|c$, then $a|c$.

Theorem 17. 1 is odd.

Theorem 18. 1 is not even.

Theorem 19. For all integers n, the relation $\equiv_n$ is reflexive, symmetric, and transitive. Specifically:

- For all integers a, $a \equiv a \pmod{n}$.
- For all integers a and b, if $a \equiv b \pmod{n}$, then $b \equiv a \pmod{n}$.
- For all integers a, b, c, if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

Theorem 20. For every integer n, n is even if and only if n^2 is even.

Theorem 21. n is even if and only if $n - 1$ is odd.

Theorem 22. For every integer n, either n is odd or n is even (but not both).

Theorem 23. $\sqrt{2}$ is irrational.

Theorem 24. For all integers a, b, c, d, m, and n, if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then:

- $a + c \equiv b + d \pmod{n}$,
- $ac \equiv bd \pmod{n}$, and
- $a^m \equiv b^m \pmod{n}$ if $m \geq 0$.