A List of Topics for the Final

Here's a list of things you should know for the final exam.

Proof Techniques

1. Ways to prove $P \to Q$.
   (a) Direct proof: assume P is true and prove that Q is true.
   (b) Proof by contrapositive: assume Q is false and prove that P is false.
   (c) Proof by contradiction: assume P is true and Q is false, and then deduce a contradiction.
   (d) Proof by cases: to show $(P \vee Q) \to R$, show that $P \to R$ and $Q \to R$.
2. Inductive proofs
   (a) Basic induction: prove the base case, then the inductive step: $P(n) \to P(n+1)$.
   (b) Strong induction: prove the base case, and then prove: $(\forall k \leq n, P(k)) \to P(n+1)$.
3. Proofs involving sets
   (a) Element-chasing: to prove $A \subseteq B$, assume $x \in A$, then prove $x \in B$.
   (b) To prove $A = B$, prove $A \subseteq B$ and $B \subseteq A$.
   (c) To prove $A = \emptyset$, assume $x \in A$ and then deduce a contradiction.
4. Proofs involving functions
   (a) To prove that $f : A \to B$ is injective, assume $f(x) = f(y)$, and prove $x = y$.
   (b) To prove that $f : A \to B$ is surjective, assume $y \in B$, and show that there must exist an x such that $f(x) = y$.
   (c) To prove f is bijective, prove that f is injective and surjective.
5. Proofs involving Countability
   (a) To prove $A \approx B$, find a bijection $f : A \to B$. Or find two injections $f : A \to B$ and $g : B \to A$.
   (b) To prove A is countable, find a bijection $f : \mathbb{N} \to A$. Equivalently, arrange the elements of A into an infinitely long list that can be labeled with natural numbers.
   (c) To prove A is uncountable, assume there exists a list of elements of A like the above, then prove there must be an element not on that list. Or, prove that $A \approx B$ where B is some set that is known to be uncountable.
6. Miscellaneous techniques (possibly useful on a bonus problem)
   (a) Invariants: prove that no matter how you do something, some property will hold. Use that property to prove something else.
   (b) Infinite descent: prove that there's no smallest natural number with some property, and therefore there are no natural numbers with that property.

Mathematical Structures

1. Logical operators
   - $\vee$, $\wedge$, $\neg$, $\rightarrow$, $\Leftrightarrow$
   - Truth tables
   - De Morgan's laws
2. Number systems
   - Same big list of axioms and theorems that you had on the midterm
   - Parity
   - Divisibility
   - Modular arithmetic
3. Sets
   - Elements and subsets
   - Unions, intersections, complements, and set differences
   - De Morgan's laws again
   - Power sets
   - Cartesian products
   - Disjoint sets
4. Functions
   - Injections, surjections, and bijections
   - Composition
   - Inverse functions and inverse images
5. Cardinality
   - Equinumerous sets
   - Countable and uncountable sets