Math 308 M - Spring 2017
Midterm Exam Number Two
May 17, 2017

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Signature: a Section: X

1	15	15
2	12	12
3	12	12
4	12	12
5	15	15
6	24	24
7	10	10
Total	100	100

- This exam consists of SEVEN problems on SIX pages, including this cover sheet.
- Show all work for full credit.
- You may use a scientific, non-graphing, non-algebraic calculator during this exam. Other calculators and electronic device are not permitted.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you write on the back of the page, please indicate that you have done so!
- Draw a box around your final answer to each problem.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

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1. **[5 points per part]** Determine whether or not each subset of $\mathbb{R}^n$ is a subspace. (No points for just guessing. Explain!)

(a) The set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ in $\mathbb{R}^4$ such that $x_1x_2 - x_3x_4 = 0$.

$\boxed{\text{No.}}$ For example, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are in this set, but their sum $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ isn't.

(b) The set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$ such that x_2 is an integer.

$\boxed{\text{No.}}$ For example, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is in this set but $1/2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ isn't.

(c) The set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$ such that $x_1 + x_2 = x_3$ and $x_1 - x_2 = 2x_3$.

$\boxed{\text{Yes.}}$ Write the equations as $\begin{matrix} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 - 2x_3 = 0 \end{matrix}$.

The solution is the null space of $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix}$, which is a subspace

2. **[12 points]** Find a vector $\mathbf{u}$ such that $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}5\\-4\\-6\end{bmatrix}, \mathbf{u} \right\}$ is a basis for $\mathbb{R}^3$.

Reduce:

$$\begin{bmatrix} 1 & 5 & 1 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 \\ 3 & -6 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \to R_2 - 2R_1 \\ \to R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 5 & 1 & 0 & 0 \\ 0 & -14 & -2 & 1 & 0 \\ 0 & -21 & -3 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \to R_3 - \frac{3}{2}R_2 \end{array} \sim \begin{bmatrix} 1 & 5 & 1 & 0 & 0 \\ 0 & -14 & -2 & 1 & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & 1 \end{bmatrix}$$

3 pivot columns

basis is $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}5\\-4\\-6\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ $\vec{u}$

3. **[4 points per part]** Let $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. $\to R_3 - R_1$ $\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Write a basis for row(A).

$$\boxed{\left\{ \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix} \right\}}$$

(b) Write a basis for col(A).

$$\boxed{\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}}$$

(c) Write a basis for null(A).

$$A\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \vec{0} \to \begin{array}{ll} x_3 = s_1 & x_1 + s_2 = 0 \\ x_4 = s_2 & x_2 + s_1 + s_2 = 0 \end{array} \to \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-s_2\\-s_1 - s_2\\s_1\\s_2\end{bmatrix} = s_1\begin{bmatrix}0\\-1\\1\\0\end{bmatrix} + s_2\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}$$

basis: $\boxed{\left\{ \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\0\\1\end{bmatrix} \right\}}$

4. **[12 points]** Let $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 0 & 3 \\ 2 & 0 & 5 \end{bmatrix}$. Find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \to R_2 - R_1 \\ \to R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ \to R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right] \begin{array}{l} \\ \to \div(-1) \\ \to \div(-1) \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \begin{array}{l} \to R_1 - 4R_3 \\ \to R_2 - R_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -8 & 4 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \begin{array}{l} \to R_1 - R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -5 & 3 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

5. **[15 points]** Let $A = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Find all eigenvalues for A.

Triangular! So, $\boxed{\lambda = 5 \ \& \ \lambda = 4}$

w/ multiplicity 2

(b) For each of the eigenvalues you found in part (a), find a basis for the corresponding eigenspace of A.

$\lambda = 5$: basis for null space of $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$

$\lambda = 4$: basis for null space of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

6. **[24 points]** A matrix merchant shows up at your door and offers to buy some matrices from you, for 8 points each. Give him the following matrices:

(a) A matrix with eigenvalues 4 and -2 and corresponding eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\3\end{bmatrix}$.

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}$$

need $A\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}-2\\-6\end{bmatrix}$

$$A=\begin{bmatrix}4 & ?\\0 & ?\end{bmatrix}$$

$$\boxed{A=\begin{bmatrix}4 & -2\\0 & -2\end{bmatrix}}$$

(b) A matrix A such that A^6 is the identity matrix, but A, A^2, A^3, A^4, and A^5 are not.

One way: rotation matrix!

$$A=\begin{bmatrix}\cos(60^\circ) & -\sin(60^\circ)\\ \sin(60^\circ) & \cos(60^\circ)\end{bmatrix}=\boxed{\begin{bmatrix}\frac{1}{2} & \frac{-\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}}$$

This also works:

$$A=\boxed{\begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\end{bmatrix}}$$

(c) A matrix whose column space and null space are equal.

Need a $2n \times 2n$ matrix with rank n whose square is 0.

Examples: $\boxed{\begin{bmatrix}0&1\\0&0\end{bmatrix}}$, $\boxed{\begin{bmatrix}1&1\\-1&-1\end{bmatrix}}$, $\boxed{\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}}$, ...

7. **[10 points]** Find all values of c for which the matrix $\begin{bmatrix} 4 & 1 & 0 & 0 \\ 5 & 1 & 4 & 3 \\ 2 & 3 & 0 & c \\ -7 & -1 & 0 & c \end{bmatrix}$ has determinant 1.

$$\text{determinant} = -4 \begin{vmatrix} 4 & 1 & 0 \\ 2 & 3 & c \\ -7 & -1 & c \end{vmatrix} = -4\left(-c \underbrace{\begin{vmatrix} 4 & 1 \\ -7 & -1 \end{vmatrix}}_{3} + c \underbrace{\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}}_{10}\right)$$

$$= -4(7c) = 1 \rightarrow c = \boxed{\frac{-1}{28}}$$

8. **[0 points]** Finished early? Here's a sudoku.

The numbers in each dashed box must form a 2×2 matrix with determinant 0.