Math 308 L - Spring 2017
Midterm Exam Number One
April 19, 2017

Name: ______________________ Student ID no. : ______________

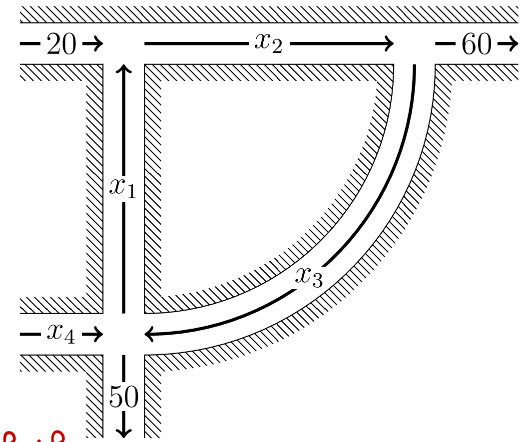
Signature: ______________________ Section: ______________

1	15	
2	12	
3	14	
4	15	
5	10	
6	12	
7	10	
8	12	
Total	100	

- This exam consists of EIGHT problems on FIVE pages, including this cover sheet.
- Show all work for full credit.
- You may use a scientific, non-graphing, non-algebraic calculator during this exam. Other calculators and electronic device are not permitted.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you write on the back of the page, please indicate that you have done so!
- Draw a box around your final answer to each problem.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

1. **[15 points]** Below is a traffic diagram of three intersections.

Find the general solution for (x_1, x_2, x_3, x_4).



traffic in = traffic out :

$$20 + x_1 = x_2 \qquad x_1 - x_2 = -20$$

$$x_2 = x_3 + 60 \rightarrow x_2 - x_3 = 60$$

$$x_3 + x_4 = x_1 + 50 \qquad -x_1 + x_3 + x_4 = 50$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -20 \\ 0 & 1 & -1 & 0 & 60 \\ -1 & 0 & 1 & 1 & 50 \end{array}\right] \xrightarrow{R_3 + R_1} \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & -1 & 1 & 1 & 30 \end{array}\right] \xrightarrow{R_3 + R_2}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -20 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 90 \end{array}\right] \xrightarrow{R_1 + R_2} \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 40 \\ 0 & 1 & -1 & 0 & 60 \\ 0 & 0 & 0 & 1 & 90 \end{array}\right] \rightarrow \boxed{\begin{aligned} x_1 &= 40 + s_1 \\ x_2 &= 60 + s_1 \\ x_3 &= s_1 \\ x_4 &= 90 \end{aligned}}$$

2. **[12 points]** Write the following matrix in **reduced** echelon form.

$$\begin{bmatrix} 1 & 3 & 5 & -9 & 2 \\ 2 & 12 & 4 & 1 & 0 \\ 0 & -2 & 1 & 3 & 9 \\ 0 & 0 & 2 & -6 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

(Found a shortcut? Great! But please explain it.)

triangular, so this part reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & -9 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 5 & -9 & 2 \end{bmatrix} \xrightarrow{R_6 - R_1 - 3R_2 - 5R_3 + 9R_4 - 2R_5} \sim \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

3. **[14 points]** Can you write the vector $\begin{bmatrix} -4 \\ 5 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 6 \\ 3 \\ 10 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$? If so, do it. If not, why not?

Let's solve: $x_1 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 6 \\ 3 \\ 10 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ 5 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 6 & -4 \\ 1 & 3 & 5 \\ 5 & 10 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 \\ 4 & 6 & -4 \\ 5 & 10 & 5 \end{bmatrix} \begin{array}{l} \\ \rightarrow R_2 - 4R_1 \\ \rightarrow R_3 - 5R_1 \end{array}$

$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -6 & -24 \\ 0 & -5 & -20 \end{bmatrix} \begin{array}{l} \\ \div(-6) \\ \div(-5) \end{array} \sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 4 \\ 0 & 1 & 4 \end{bmatrix} \begin{array}{l} \rightarrow R_1 - 3R_2 \\ \\ \rightarrow R_3 - R_2 \end{array} \sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} x_1 = -7 \\ x_2 = 4 \end{array}$

$$\begin{bmatrix} -4 \\ 5 \\ 5 \end{bmatrix} = \boxed{-7 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + 4 \begin{bmatrix} 6 \\ 3 \\ 10 \end{bmatrix}}$$

4. **[15 points]** Here are four vectors: $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ z_1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ z_2 \end{bmatrix}$.

(a) For what values of z_1 and z_2 do these vectors span $\mathbb{R}^3$?

↳ Reduce $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 4 & 2 & 0 \\ -1 & 1 & z_1 & z_2 \end{bmatrix} \begin{array}{l} \\ \rightarrow R_2 - 2R_1 \\ \rightarrow R_3 + R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & 4 & -2 \\ 0 & 2 & z_1 - 1 & z_2 + 1 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3 - R_2 \end{array}$

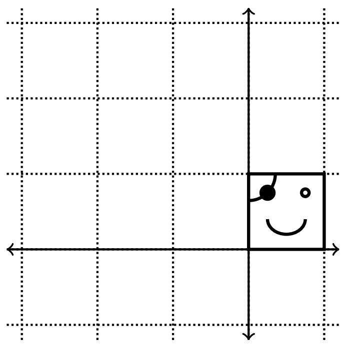
$\sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & z_1 - 5 & z_2 + 3 \end{bmatrix} \longleftarrow$ spans $\mathbb{R}^3$ if and only if this row is not all zeroes, so $\boxed{\text{unless } z_1 = 5 \text{ \& } z_2 = -3.}$

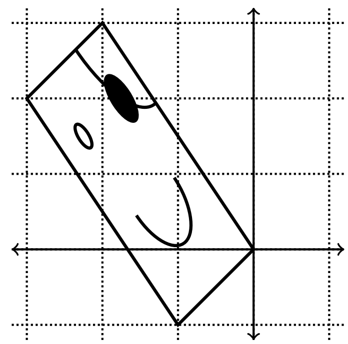
(b) For what values of z_1 and z_2 are these vectors linearly **independent**?

$\boxed{\text{None!}}$ 4 vectors in $\mathbb{R}^3$ are <u>always</u> linearly dependent.

5. **[10 points]** Below on the left is a picture of Victor, a humble unit square chilling in $\mathbb{R}^2$.

One day, a witch cursed him with a linear transformation, turning him into the parallelogram on the right!





The witch's spell was formed by applying the function $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A. What's A?

$$\boxed{\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}}$$

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ $\quad$ $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

6. **[12 points]** Find three vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ such that each of the pairs $\{\mathbf{u}_1, \mathbf{u}_2\}$, $\{\mathbf{u}_1, \mathbf{u}_3\}$, and $\{\mathbf{u}_2, \mathbf{u}_3\}$ are linearly independent, but $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are linearly **dependent**.

Any 3 non-parallel vectors that lie in a plane.

$\boxed{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \text{ and } \begin{bmatrix}1\\1\end{bmatrix},}$ for example.

7. **[10 points]** Let $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 4 \\ x_1 + x_2 \end{bmatrix}$. Is this a linear transformation? Why or why not?

$\boxed{\text{No!}}$ $T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$.

But that means $\underbrace{T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)}_{\begin{bmatrix} -4 \\ 0 \end{bmatrix}} \neq \underbrace{T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)}_{\begin{bmatrix} -8 \\ 0 \end{bmatrix}}$

8. **[3 points per part]** Here, I bought you this linear transformation:

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

Answer the following questions. Explain your reasoning!

(a) What's the domain of T?

$\boxed{\mathbb{R}^4}$ because there are 4 columns.

(b) What's the codomain of T?

$\boxed{\mathbb{R}^3}$ because there are 3 rows.

(c) Is T one-to-one?

$\boxed{\text{No!}}$ Four column vectors in $\mathbb{R}^3$ cannot be linearly independent.

(d) Is T onto?

$\boxed{\text{Yes!}}$ The column vectors span $\mathbb{R}^3$. (You can tell because the matrix is in echelon form and there's no row of all zeroes.)