Math 300 B - Winter 2017
Midterm Exam
February 1, 2017

Name: ______________________ Student ID no. : ______________

Signature: ______________________ Section: __________

1	5	
2	5	
3	12	
4	12	
5	6	
Total	40	

- This exam consists of FIVE problems on SIX pages, including this cover sheet and a page of axioms and theorems.
- No calculators or other electronic devices are allowed (or needed).
- All manipulations of integer expressions must be justified using the axioms and elementary properties of the integers provided on the last page.
- You may use the last page for scratch work. Anything you want me to read should be written below the appropriate problem.
- You have 50 minutes to complete the exam.

1. **[5 points]** Write a meaningful negation of the following statements.

(a) "For all integers $n > 2$, $2^n - 1$ is composite or $2^n + 1$ is composite."
("Composite" means "not prime".)

There exists an integer $n>2$ such that 2^n-1 is prime and 2^n+1 is prime.

(b) "For all integers n, if n is odd, then there exists an integer k such that $n^2 - 1 = 4k$."

There exists an integer n such that n is odd and for all integers k, $n^2-1 \neq 4k$.

2. **[5 points]** Let $S = \{a, b, c, d\}$. Consider the following relation on S:

$$R = \{(a, a), (a, b), (a, c), (b, b), (b, d), (c, b), (d, c)\}$$

(a) Is R asymmetric?

No! (a,a) (and (a,a)) are elements of R.

(b) Is R transitive?

No! For example, $(a,b) \in R$ and $(b,d) \in R$, but $(a,d) \notin R$.

3. **[12 points]** Suppose n is an integer. Prove that if $4|n-3$, then $4|n^2+n$.

Suppose $4|n-3$.

Then by the definition of divisibility, there exists an integer k such that $4k = n-3$.

By Axiom 2, $4k+4 = (n-3)+4$.

By Axiom 3, $4k+4 = n+(-3+4) = n+1$.

By Axiom 5, $4(k+1) = n+1$.

By Axiom 2, $4(k+1)n = (n+1)n$.

Again applying Axiom 5: $4(k+1)n = n^2+n$.

And by the Axiom 1, $(k+1)n$ is an integer.

So $n^2+n = 4\cdot(\text{some integer})$, which means $4|n^2+n$.

4. **[12 points]** Let A, B, and C be sets.

Prove that $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$.

We will show inclusion in both directions.

$(\subseteq)$: Let $x \in (A \setminus B) \setminus C$.

So $x \in A \setminus B$ and $x \notin C$.

Since $x \in A \setminus B$, we also know $x \in A$ and $x \notin B$.

Since $x \in A$ and $x \notin C$, $x \in A \setminus C$.

And since $x \notin B$, $x \notin B \setminus C$.

So $x \in (A \setminus C) \setminus (B \setminus C)$.

So $(A \setminus B) \setminus C \subseteq (A \setminus C) \setminus (B \setminus C)$.

$(\supseteq)$: Let $x \in (A \setminus C) \setminus (B \setminus C)$.

So $x \in A \setminus C$ and $x \notin B \setminus C$.

Since $x \in A \setminus C$, we know $x \in A$ and $x \notin C$.

Now, is it possible that $x \in B$?

No, because then we would have $x \in B \setminus C$, which is false.

So $x \in A$ and $x \notin B$, which means $x \in A \setminus B$.

And $x \notin C$, so $x \in (A \setminus B) \setminus C$.

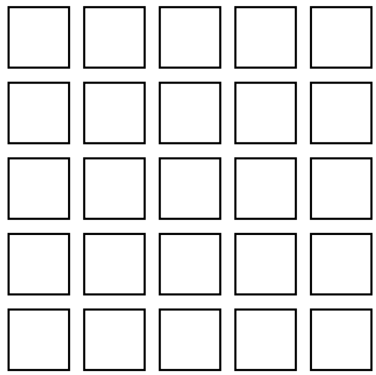
So $(A \setminus C) \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C$.

We've shown containment in both directions, so $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$.

5. **[6 points]** Suppose a 5×5 grid of boxes is filled with integers in such a way that every number is equal to the average of its neighbors.

(The neighbors of a box are the ones that share an edge with that box. For example, a corner box has two neighbors, while a central box has four neighbors.)

Prove that all 25 boxes must contain the same integer.



Suppose not all numbers are the same.

Let the smallest number be x.

There must be some box containing x adjacent to a box with a larger number.

Say the x-box has k neighbors, $a_1, a_2, \ldots, a_k$, where $a_i \geq x$ for all i, and $a_k > x$.

the larger number

x is the average of its neighbors:

$$x = \frac{a_1 + a_2 + \cdots + a_k}{k}$$

$$\downarrow$$

$$\overbrace{x + \cdots + x + x}^{k \text{ times}} = a_1 + a_2 + \cdots + a_k$$

But by the inequalities above $x + x + \cdots + x \leq a_1 + \cdots + a_{k-1} + x < a_1 + \cdots + a_k$, which contradicts the equality. So all numbers are equal.

Known Axioms and Theorems

You may use any of the following axioms and theorems. If you do, please cite them by number.

Axioms of the Integers

Suppose a, b, and c are integers.

1. $a + b$ and ab are integers.
2. If $a = b$, then $a + c = b + c$ and $ac = bc$.
3. $a + b = b + a$ and $ab = ba$.
4. $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
5. $a(b + c) = ab + ac$.
6. $a + 0 = 0 + a = a$, and $a \cdot 1 = 1 \cdot a = a$.
7. There exists an integer $-a$ such that $a + (-a) = (-a) + a = 0$.
8. Exactly one of the following is true: $a < 0$, $a > 0$, or $a = 0$.
9. $a \leq a$.
10. If $a \leq b$ and $b \leq a$, then $a = b$.
11. If $a \leq b$ and $b \leq c$, then $a \leq c$.
12. If $a \leq b$, then $a + c \leq b + c$.

    Also, if $c \geq 0$, then $ac \leq bc$.
13. $0 < 1$.
14. If S is a nonempty set of positive integers, then S has a minimal element.

Elementary Properties of the Integers

Suppose a, b, c, and d are integers.

1. $a \cdot 0 = 0$
2. If $a + c = b + c$, then $a = b$.
3. $-a = (-1) \cdot a$
4. $(-a) \cdot b = -(a \cdot b)$
5. $(-a) \cdot (-b) = a \cdot b$
6. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

11. If $a < b$ and $c < 0$, then $bc < ac$.
12. If $a < b$ and $c < d$, then $a + c < b + d$.
13. If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
14. If $a < b$, then $-b < -a$.
15. $0 \leq a^2$.
16. If $ab = 1$, then either $a = b = 1$ or $a = b = -1$.

Note 1: Properties 7 through 10 from the textbook are skipped because we're using them as axioms.

Note 2: Properties 11 through 14 still hold if each $<$ is replaced by a $\leq$.