DFEP #1: Friday, January 13th.

Write each limit as an integral, then compute it.

(a) $\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{3n}\right)\frac{\pi}{3n}$

(b) Warning: not easy! $\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-\frac{i^2}{4}}}$

DFEP #1 Solution:

(a) We want to write $\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{3n}\right)\frac{\pi}{3n}$ as an integral. Recall that using a limit of right-hand Riemann sums, we have $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$, where $\Delta x = \frac{b-a}{n}$.

Stare at this limit a bit and you can see it's what you get when $a=0$, $b=\frac{\pi}{3}$, and $f(x)=\cos(x)$.

So it's

$$\int_0^{\pi/3}\cos(x)\,dx = \sin(x)\Big]_0^{\pi/3} = \sin\left(\frac{\pi}{3}\right)-\sin(0)=\frac{\sqrt{3}}{2}.$$

(b) Hey, I warned you, this one is tough. First, a little algebra:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-\frac{i^2}{4}}} = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n\sqrt{1-\left(\frac{i}{2n}\right)^2}}$$

That $\frac{i}{2n}$ sure looks like it wants to be x_i, which means Δx should be $\frac{1}{2n}$. So rewrite it as:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{2}{\sqrt{1-\left(\frac{i}{2n}\right)^2}}\cdot\frac{1}{2n}$$

This is the limit of left-hand Riemann sums for the integral

$$\int_0^{\frac{1}{2}}\frac{2}{\sqrt{1-x^2}}\,dx = 2\arcsin(1/2)-2\arcsin(0)=\frac{\pi}{3}$$

DFEP #2: Wednesday, January 18th.

Consider the function $f(x)=\int_{\sin(2x)}^{3x} e^{t^2}\,dt$.

Is $f(x)$ concave up or concave down at $x=\frac{\pi}{2}$?

DFEP #2 Solution:

Okay, we need the second derivative of $f(x) = \displaystyle\int_{\sin(2x)}^{3x} e^{t^2}\,dt$.

Let $g(x) = \int_0^x e^{t^2}\,dt$. According to the fundamental theorem of calculus, $g'(x) = e^{x^2}$. Furthermore, $f(x) = g(3x) - g(\sin(2x))$. So:

$$f'(x) = 3g'(3x) - 2\cos(2x)g'(\sin(2x)) = 3e^{(3x)^2} - 2\cos(2x)e^{\sin^2(2x)}$$

Differentiating again gives:

$$f''(x) = 54xe^{(3x)^2} + 4\sin(2x)e^{\sin^2(2x)} - 8\cos^2(2x)\sin(2x)e^{\sin^2(2x)}$$

And therefore $f''\left(\frac{\pi}{2}\right) = 54\left(\frac{\pi}{2}\right)e^{(3\pi/2)^2} > 0$, so the function is concave up.

DFEP #3: Friday, January 20th.

Your train leaves New York for Philadelphia at 9:00 AM at a speed of 100 miles per hour. Seated next to you is a man staring at a page of tricky integrals. Solve the integrals for him.

(a) $\displaystyle\int \frac{1}{x\ln(x^2)}\,dx$

(b) $\displaystyle\int e^{e^x+x}\,dx$

(c) $\displaystyle\int_0^3 x^5\sqrt[3]{x^2-16}\,dx$

DFEP #3 Solution:

(a) Set $u = \ln(x^2)$, $du = \dfrac{2x}{x^2}\,dx = \dfrac{2}{x}\,dx$.

So $\displaystyle\int \frac{1}{x\ln(x^2)}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|\ln(x^2)| + C.$

(b) Rewrite as $\displaystyle\int e^x e^{e^x}\,dx$. Then $u = e^x$, $du = e^x\,dx$ gives:

$\displaystyle\int e^u\,du = e^u + C = e^{e^x} + C.$

(c) Let $u = x^2 - 16$, so $x^4 = (u+16)^2$ so the integral becomes

$$\frac{1}{2}\int_{-16}^{-7} (u+16)^2\sqrt[3]{u}\,du$$

Expand to get

$$\frac{1}{2}\int_{-16}^{-7} \left(u^{7/3} + 32u^{4/3} + 256u^{1/3}\right)\,du$$

which is solved easily enough by the power rule:

$$\frac{1}{2}\left[\frac{3u^{10/3}}{10} + \frac{96u^{7/3}}{7} + 192u^{4/3}\right]_{-16}^{-7} \approx -254.1$$

DFEP #4: Monday, January 23rd.

Find a such that the area of the region in the first quadrant bounded by $y = x^3$ and $y = ax$ is 5.

DFEP #4 Solution:

What are the bounds on this region? $x^3 = ax$ means that $x = 0$, $x = \sqrt{a}$, or $x = -\sqrt{a}$. The region in the first quadrant runs from 0 to $\sqrt{a}$, and over that interval, $ax \geq x^3$

So the area in question is

$$\int_0^{\sqrt{a}} (ax - x^3)\, dx = \frac{ax^2}{2} - \frac{x^4}{4}\Big]_0^{\sqrt{a}} = \frac{a^2}{4} = 5$$

So $a = \sqrt{20}$.

DFEP #5: Friday, January 27th.

Compute the area of the solid formed by rotating the region between $y = e^x$ and $y = \dfrac{x-1}{x}$ from $x = 1$ to $x = 4$ around the line $y = -1$.

DFEP #5 Solution:

On the interval $[1, 4]$, we know that $\dfrac{x-1}{x} < 1 < e \le e^x$, so $y = \dfrac{x-1}{x}$ is always below $y = e^x$. That means the volume is given by:

$$\int_1^4 \pi\left((e^x+1)^2 - \left(\frac{x-1}{x}+1\right)^2\right)\,dx$$

This isn't too bad. (To integrate e^{2x}, just use $u = 2x$.)

$$\begin{aligned}
&\pi\int_1^4 \left(e^{2x} + 2e^x + 1 - \frac{4x^2-4x+1}{x^2}\right)dx \\
&= \pi\int_1^4 \left(e^{2x} + 2e^x + 1 - 4 - \frac{4}{x} + \frac{1}{x^2}\right)dx \\
&= \pi\left(\frac{1}{2}e^{2x} + 2e^x - 3x - 4\ln|x| - \frac{1}{x}\right)\Big]_1^4 \\
&= \pi\left(\frac{1}{2}e^8 + 2e^4 - 12 - 4\ln|4| - \frac{1}{4}\right) - \pi\left(\frac{1}{2}e^2 + 2e - 3 - 0 - \frac{1}{1}\right)
\end{aligned}$$

DFEP #6: Monday, January 30th.

A hemispherical pot with diameter 50 cm is filled to the brim with tomato soup of uniform density 1500 kg/m^3. Find the work required to drink all of the soup with a straw. (The top of the straw is level with the rim of the tank.)

DFEP #6 Solution:

Let's imagine cutting the soup into horizontal cross sections (it's a very thick soup, I guess): how much work does it take to bring each cross section to the top?

Say $y = 0$ is the bottom of the pot and $y = 0.25$ is the top of the pot. (We're working in meters.) At a depth of y meters from the top, we get a slice with area $\pi\left(\sqrt{.25^2 - y^2}\right)^2 = \pi(.25^2 - y^2)$ (hint: to see this, use the Pythagorean theorem). So the mass at that height is $\pi(.25^2 - y^2)(1500)\,\Delta y$, and the work required is $9.8 \cdot 1500\pi(.25^2 - y^2)y\,\Delta y$.

So the total work required is

$$\begin{aligned}
&\int_0^{0.25} 9.8 \cdot 1500\pi(.25^2 - y^2)y\,dy \\
&= 9.8 \cdot 1500\pi \int_0^{0.25} \left(-y^3 + \frac{y}{16}\right) dy \\
&= 9.8 \cdot 1500\pi \left(\frac{-y^4}{4} + \frac{y^2}{32}\right)\Bigg]_0^{0.25} \\
&= 9.8 \cdot 1500\pi \left(\frac{-1}{1024} + \frac{1}{512}\right) \\
&= \frac{9.8 \cdot 1500\pi}{1024} \text{ J}
\end{aligned}$$

(Note: a previous version of this solution was missing a factor of 9.8, and had the wrong radius.)

DFEP #7: Wednesdays, February 1st.

Determine the average value of the function $f(x) = e^{2x}(x^2 - 5x + 3)$ on the interval $[0, 2]$.

DFEP #7 Solution:

Okay, we want the average value of $f(x) = e^{2x}(x^2 - 5x + 3)$ on the interval $[0, 2]$, so we want to find $\frac{1}{2}\int_0^2 e^{2x}(x^2 - 5x + 3)\,dx$. Let's use integration by parts! $u = x^2 - 5x + 3$, $dv = e^{2x}\,dx$, so $du = (2x - 5)\,dx$, and $v = \frac{1}{2}e^{2x}$. So:

$$\frac{1}{2}\int_0^2 e^{2x}(x^2 - 5x + 3)\,dx = \frac{1}{2}\left(\frac{1}{2}e^{2x}(x^2 - 5x + 3)\Big]_0^2 - \frac{1}{2}\int_0^2 e^{2x}(2x - 5)\,dx\right)$$

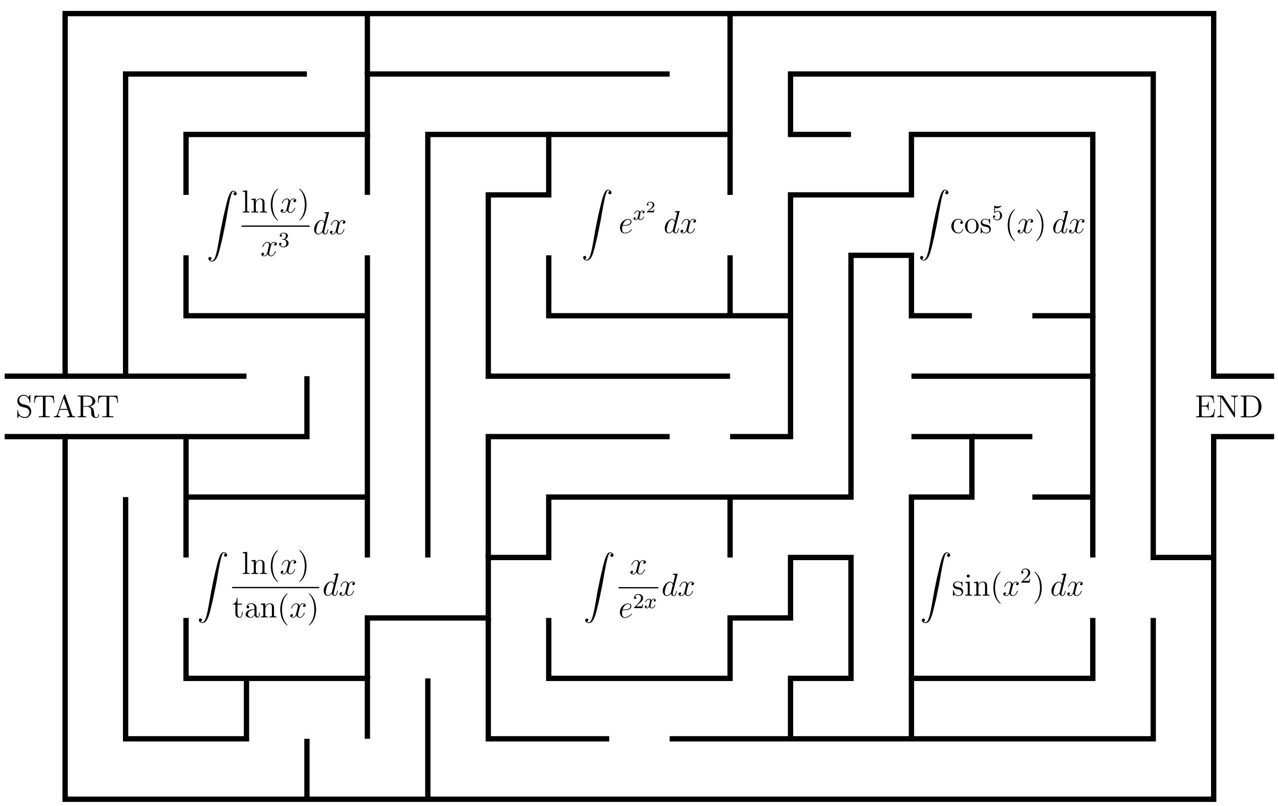
Again! $u = (2x - 5)$, $dv = e^{2x}$, etc:

$$= \frac{1}{2}\left(\left(\frac{1}{2}(e^{2x}(x^2 - 5x + 3)) - \frac{1}{4}e^{2x}(2x - 5)\right)\Big]_0^2 + \frac{1}{2}\int_0^2 e^{2x}\,dx\right)$$

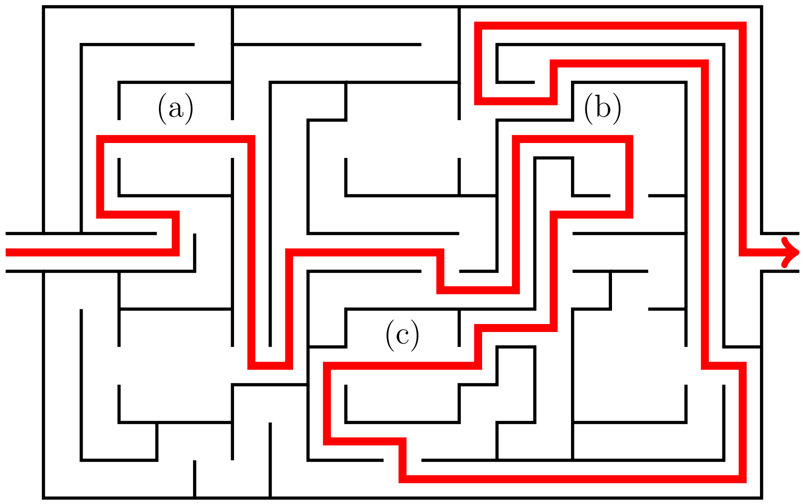
This is fun, right? Anyway you get $-3 - e^4$.

DFEP #8: Friday, February 3rd.

Complete the following maze. But be warned: you may only pass through a room with an integral if you can evaluate it, and some of these integrals are impossible!



DFEP #8 Solution:



(a) $\int \frac{\ln(x)}{x^3} dx$. Let $u = \ln(x)$, $dv = x^{-3} dx$, then this becomes $-\frac{\ln(x)}{2x^2} + \int \frac{1}{2x^3} dx$, or $-\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} + C$.

(b) $\int \cos^5(x)\,dx = \int \cos(x)(1-\sin^2(x))^2\,dx = \int (1-u^2)^2\,du = \int (1-2u^2+u^4)\,du$.
Integrate and resubstitute to get $\sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C$.

(c) $\int \frac{x}{e^{2x}} dx$. Let $u = x$, $dv = e^{-2x} dx$, so $du = dx$, $v = \frac{-1}{2} e^{-2x}\,dx$ and we get $\frac{-x}{2e^{2x}} + \int \frac{e^{-2x}}{2}\,dx = \frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}} + C$.

DFEP #9: Wednesday, February 8th.

Let $\mathcal{R}$ be the region bounded by $y = 0$, $x = 0$, $x = 4$, and $y = \sqrt{x^2 + 6x + 25}$. Compute the volume of the solid formed by revolving $\mathcal{R}$ around the y-axis.

DFEP #9 Solution:

Using the shell method, we want $\int_0^4 2\pi x\sqrt{x^2+6x+25}\,dx$. Let's complete the square to get

$$2\pi\int_0^4 x\sqrt{(x+3)^2+16}\,dx$$

Setting $u = x+3$, gives $2\pi\int_3^7 (u-3)\sqrt{u^2+16}\,du$, and now we can use trigonometric substitution with $u = 4\tan(\theta)$, $du = 4\sec^2(\theta)d\theta$, leaving the integral

$$2\pi\int_{\arctan(3/4)}^{\arctan(7/4)} (4\tan(\theta)-3)\sqrt{4\sec^2\theta}4\sec^2\theta\,d\theta$$

This simplifies to:

$$\begin{aligned}&16\pi\int_{\arctan(3/4)}^{\arctan(7/4)} (4\tan(\theta)\sec^3(\theta) - 3\sec^3(\theta))\,d\theta \\ &= 16\pi\left(4\sec^3(\theta)/3 - \frac{3}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right)\right)\Bigg]_{\arctan(3/4)}^{\arctan(7/4)}\end{aligned}$$

I can't really blame you if you don't want to write all that out.

DFEP #10: Friday, February 10th.

Compute the area of the region bounded by the curves $y = 2x$, $y = \dfrac{2x^3+10x^2+16x+7}{x^2+5x+6}$, $x = 0$, and $x = 2$.

DFEP #10 Solution:

We want $\displaystyle\int_0^2 \left(\frac{2x^3+10x^2+16x+7}{x^2+5x+6} - 2x\right)\,dx = \int_0^2 \frac{4x+7}{(x+1)(x+5)}\,dx$. Okay, cool, let's do partial fractions:

$$\begin{aligned}
&\int_0^2 \left(\frac{3/4}{x+1} + \frac{13/4}{x+5}\right)\,dx \\
&= \left(\frac{3}{4}\ln|x+1| + \frac{13}{4}\ln|x+5|\right)\Big]_0^2 \\
&= \frac{3}{4}\ln(3) + \frac{13}{4}\ln(7) - \frac{13}{4}\ln(6)
\end{aligned}$$

DFEP #11: Monday, February 13th.

Jeannette is three years older than Hortense, and in two years Hortense will be twice as old as Bertrand. Bertrand is taller than Hortense, but shorter than Jeanette.

Anyway, here are some integrals.

(a) $\displaystyle\int \frac{4x^4+10x^3+9x^2+16x+8}{x^3+2x^2+x}dx$

(b) $\displaystyle\int \sin^5(x)\cos^8(x)\,dx$

(c) $\displaystyle\int \frac{\sqrt{16+x^2}}{x}dx$

DFEP #11 Solution:

(a) Long division and partial fractions yields

$$\begin{aligned}&\int\left(\frac{5}{(x+1)^2}-\frac{7}{x+1}+\frac{8}{x}+4x+2\right)dx\\ =&\frac{-5}{x+1}-7\ln|x+1|+8\ln|x|+2x^2+2x+C\end{aligned}$$

(b) First, rewrite as:

$$\int \sin(x)\left(\sin^2(x)\right)^2\cos^8(x)\,dx=\int \sin(x)\left(1-\cos^2(x)\right)^2\cos^8(x)\,dx$$

Then use $u=\cos(x)$, $du=-\sin(x)\,dx$ to get:

$$\int -(1-u^2)^2u^8\,du=\int(-u^{12}+2u^{10}-u^8)\,du=-\frac{u^{13}}{13}+\frac{2u^{11}}{11}-\frac{u^9}{9}+C$$

and resubstitute:

$$-\frac{\cos^{13}(x)}{13}+\frac{2\cos^{11}(x)}{11}-\frac{\cos^9(x)}{9}+C$$

(c) We want $\int\frac{\sqrt{16+x^2}}{x}dx$. Looks like a job for trigonometric substitution, no?

Let $x=4\tan(\theta)$ and the integral simplifies to $\int\frac{4\sec^3(\theta)}{\tan(\theta)}d\theta$. Use the identity $\sec^2(\theta)=1+\tan^2(\theta)$ and this simplifies to

$$\int(\csc(\theta)+\sec(\theta)\tan(\theta))\,d\theta=4\left(\ln|\csc(\theta)-\cot(\theta)|+\sec(\theta)\right)+C$$

Finally, using a comparison triangle to resubstitute:

$$4\left(\ln\left|\frac{\sqrt{x^2+16}}{x}-\frac{4}{x}\right|+\frac{\sqrt{x^2+16}}{4}\right)+C$$

DFEP #12: Friday, February 17th.

Use Simpson's Rule with $n = 6$ to estimate the average value of $f(x) = \sqrt{x^3 + 5}$ on the interval $[-1, 11]$.

DFEP #12 Solution:

So we want to estimate $\frac{1}{12}\int_{-1}^{11} \sqrt{x^3+5}\,dx$ using Simpson's Rule with $n = 6$. That means $\Delta x = 2$, so we have:

$$\frac{1}{12} \cdot \frac{2}{3}\left(\sqrt{4} + 4\sqrt{6} + 2\sqrt{32} + 4\sqrt{130} + 2\sqrt{348} + 4\sqrt{734} + \sqrt{1336}\right)$$

DFEP #13: Friday, February 24th.

Determine whether the following integral is convergent or divergent. (You do not need to evaluate the integral.)

$$\int_0^\infty \frac{dx}{\sqrt{x^3+5}}$$

DFEP #13 Solution:

Well, first off, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^3+5}}$ converges because it's just a regular definite integral. What about the rest? On the interval $[1, \infty)$, we know that $\sqrt{x^3+5} > \sqrt{x^3}$, so

$$\frac{1}{\sqrt{x^3+5}} < \frac{1}{x^{3/2}}.$$

Since $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$ converges by the p-test, so does this.

DFEP #14: Monday, February 27th.

Compute $\displaystyle\int_2^5 \left(\ln(x-2) + \frac{1}{\sqrt{5-x}}\right)\, dx.$

DFEP #14 Solution:

$\int_2^5 \left(\ln(x-2) + \frac{1}{\sqrt{5-x}}\right)\, dx$ is an improper integral for two reasons: $\ln(x-2)$ has an asymptote at 2, and $\frac{1}{\sqrt{5-x}}$ has an asymptote at 5. So let's break it into two pieces:

$$\int_2^5 \left(\ln(x-2) + \frac{1}{\sqrt{5-x}}\right)\, dx = \lim_{t\to 2^+}\left(\int_t^5 \ln(x-2)\, dx\right) + \lim_{t\to 5^-}\left(\int_2^t \frac{1}{\sqrt{5-x}}\, dx\right)$$

For the first integral, we use integration by parts and end up with:

$$\lim_{t\to 2^+}\big((x-2)(\ln(x-2)-1)\big)\Big]_t^5 = 3\ln(3) - 3 - \lim_{t\to 2^+}\big((t-2)(\ln(t-2)-1)\big)$$

l'Hôpital's rule tells us that the limit is zero, so this part is just $3\ln(3)-3$. The other part is:

$$\lim_{t\to 5^-}\left(-2\sqrt{5-x}\right)\Big]_2^t = 2\sqrt{3}$$

So in total, the integral is $3\ln(3) - 3 + 2\sqrt{3}$.

DFEP #15: Friday, March 3rd:

Solve the differential equation $y' = xy\sin(x)$ with initial condition $y(\pi) = 1$.

DFEP #15 Solution:

We want to solve $\frac{dy}{dx} = xy\sin(x)$ with $y(\pi) = 1$. Separate variables to get

$$\frac{dy}{y} = x\sin(x)\,dx.$$

Integrate (using integration by parts on the right side) to get

$$\ln|y| = -x\cos(x) + \sin(x) + C$$

Plug in $x = \pi$, $y = 1$ to get $0 = \pi + C$, so $C = -\pi$, and we have the equation

$$\ln|y| = -x\cos(x) + \sin(x) - \pi$$

We want the continuous piece of this curve containing $(\pi, 1)$, so $y > 0$ and we have

$$y = e^{-x\cos(x)+\sin(x)-\pi}.$$

DFEP #16: Monday, March 6th.

Westley has a jug that contains 4 liters of wine with 70 grams of Iocaine powder mixed in. At time $t = 0$, he begins pouring in more wine and Iocaine powder at a rate of 0.5 liters of wine and 2 grams of powder per minute. At the same time, the jug is mixed well and 0.5 liters of the mixture are poured out per minute (and safely disposed of).

1 gram of Iocaine powder is fatal to the average person, but Westley can withstand twice that amount. How long should Westley keep mixing the wine in this way so that he can safely drink a 200 mL glass, but no one else can?

Express your answer as an interval of time, e.g.: "He should stop between 5 and 7 minutes".