Math 126 F - Spring 2016
Midterm Exam Number One
April 21, 2016

Name: ____________________ Student ID no. : ______________

Signature: ______________________________ Section: ____________

1	15	
2	15	
3	8	
4	7	
5	8	
6	7	
Total	60	

- This exam consists of SIX problems on FIVE pages, including this cover sheet.
- Show all work for full credit. Show no work for zero credit.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- Write all of your work on the exam itself. If you use the back of the page, please indicate that you have done so!
- You may use a TI-30X IIS on this exam. No other electronic devices are allowed.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

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1. Consider the space curve defined by the vector function $\mathbf{r}(t) = \langle 2t^3 + 1, 3t + 4, 3t^3 \rangle$.

   (a) **[7 points]** Find the point where the space curve intersects the plane $3x + 2y - 2z = -7$.

$$3(2t^3+1) + 2(3t+4) - 2(3t^3) = -7$$

$$6t^3 + 3 + 6t + 8 - 6t^3 = -7$$

$$6t = -18$$

$$t = -3$$

plug into $\vec{r}(t)$:

$$x = 2(-3)^3 + 1 = -53$$

$$y = 3(-3) + 4 = -5$$

$$z = 3(-3)^3 = -81$$

$$\boxed{(-53, -5, -81)}$$

   (b) **[8 points]** At the point from part (a), compute the angle between the plane and the tangent line to the curve. Give your answer in degrees.

First, find $\vec{r}\,'(t) = \langle 6t^2, 3, 9t^2 \rangle$

$\vec{r}\,'(-3) = \langle 54, 3, 81 \rangle$

Next, find angle between $\langle 54, 3, 81 \rangle$ and normal vector of plane $\langle 3, 2, -2 \rangle$:

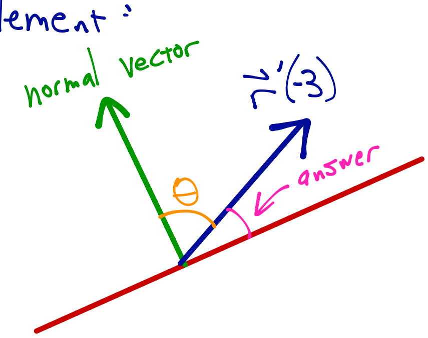
$$\underbrace{\vec{a}\cdot\vec{b}}_{6} = \underbrace{|\vec{a}|}_{\sqrt{9486}}\ \underbrace{|\vec{b}|}_{\sqrt{17}} \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{17}\sqrt{9486}}\right) = 89.144^\circ$$

But we want the complement:

$$90^\circ - 89.144^\circ$$

$$= \boxed{0.856^\circ}$$



2. Let ℓ_1 be the line $\dfrac{x+5}{-3} = y - 2 = \dfrac{z-1}{4}$.

(a) **[7 points]** Find the equation of the plane containing ℓ_1 and the origin.

Plane contains: ① Direction vector of line: $\langle -3, 1, 4 \rangle$

② Vector from origin to point on line: $\langle -5, 2, 1 \rangle$

Use × product as normal: $\langle -7, -17, -1 \rangle$

Plane with normal vector $\langle -7, -17, -1 \rangle$, containing $(0,0,0)$:

$$\boxed{-7x - 17y - z = 0}$$

(b) **[8 points]** Let ℓ_2 be the line with parametric equations $x = 7+9t$, $y = 5-t$, $z = -1-8t$. Are lines ℓ_1 and ℓ_2 parallel, intersecting, or skew?

Direction vectors: $\langle -3, 1, 4 \rangle$ & $\langle 9, -1, -8 \rangle$ ← not parallel: $\frac{-3}{9} \neq \frac{1}{-1}$

Intersecting?

$x = -5 - 3s = 7 + 9t \rightarrow s = -4 - 3t$

$y = 2 + s = 5 - t \rightarrow 2 + (-4 - 3t) = 5 - t \rightarrow -7 = 2t$

$z = 1 + 4s = -1 - 8t$

Check!

$t = -3.5$

$s = 6.5$

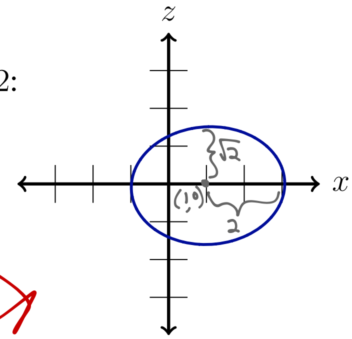
$1 + 4(6.5) \stackrel{?}{=} -1 - 8(-3.5)$

$27 = 27$

Yes! $\boxed{\text{Intersecting!}}$

3. Here's a surface: $x^2 - 2x - y^2 + 8y + 2z^2 = 15$

(a) **[4 points]** Sketch the trace of this surface in the plane $y = 2$:



$$x^2 - 2x - 4 + 16 + 2z^2 = 15$$

$$x^2 - 2x + 1 \quad + 2z^2 = 3 \quad + 1$$

$$(x-1)^2 + 2z^2 = 4$$

$$\frac{(x-1)^2}{2^2} + \frac{z^2}{(\sqrt{2})^2} = 1$$

ellipse!

(b) **[4 points]** Identify the quadric surface.

$$x^2 - 2x + 1 - y^2 + 8y - 16 + 2z^2 = 15 + 1 - 16$$

$$(x-1)^2 - (y-4)^2 + 2z^2 = 0$$

That's an $\boxed{\text{elliptical cone!}}$

("cone" is fine.)

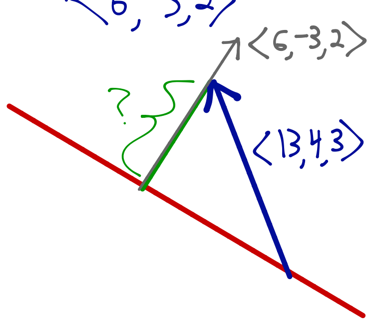
4. **[7 points]** Find the distance from the point $(14, 4, 4)$ to the plane $6x - 3y + 2z = 8$.

1) Pick some point on the plane P: $(1,0,1)$

2) Find a vector from P to $(14,4,4)$: $\langle 13, 4, 3 \rangle$.

3) Find component of $\langle 13,4,3 \rangle$ along normal vector to plane!

$$\text{comp}_{\langle 6,-3,2 \rangle} \langle 13,4,3 \rangle = \frac{\langle 6,-3,2 \rangle \cdot \langle 13,4,3 \rangle}{\| \langle 6,-3,2 \rangle \|} = \frac{78 - 12 + 6}{\sqrt{36+9+4}}$$



$$= \boxed{\frac{72}{7}}$$

5. **[8 points]** Consider the tomato-shaped polar curve $r = 1 - \sin(\theta)$.

Find the equation for the tangent line to this curve at the point where $\theta = \dfrac{\pi}{4}$.

We'll need r, $\dfrac{dr}{d\theta}$, $\dfrac{dx}{dy}$, x, and y.

$$r = 1 - \sin\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2}$$

$$x = r\cos\theta = \left(1 - \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}-1}{2}$$

$$y = r\sin\theta = \left(1 - \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}-1}{2}$$

$$\frac{dr}{d\theta} = -\cos(\theta) = -\cos\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\left(\frac{-\sqrt{2}}{2}\right)\cancel{\left(\frac{\sqrt{2}}{2}\right)} + \left(1 - \frac{\sqrt{2}}{2}\right)\cancel{\left(\frac{\sqrt{2}}{2}\right)}}{\left(\frac{-\sqrt{2}}{2}\right)\cancel{\left(\frac{\sqrt{2}}{2}\right)} - \left(1 - \frac{\sqrt{2}}{2}\right)\cancel{\left(\frac{\sqrt{2}}{2}\right)}} = \frac{1-\sqrt{2}}{-1} = \sqrt{2}-1$$

$$\boxed{y = \left(\sqrt{2}-1\right)\left(x - \frac{\sqrt{2}-1}{2}\right) + \frac{\sqrt{2}-1}{2}}$$

6. **[7 points]** Consider the space curve for the vector function $\mathbf{r}(t) = \langle \underbrace{\sin^2(t) + \sin(t)}_{\text{period } 2\pi}, \underbrace{\cos(t/2)}_{\text{period } 4\pi} \rangle$.

Write (but, seriously, do *not* evaluate) an integral to compute the length of the curve.

(Hint: The bounds are not $-\infty$ and ∞. $\mathbf{r}(t)$ is periodic.)

To find bounds, check period of components: period 2π period 4π.

So the space curve is a loop from $t=0$ to $t=4\pi$.

$$\text{Length} = \int_0^{4\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$\boxed{= \int_0^{4\pi} \sqrt{\left(2\sin t\cos t + \cos t\right)^2 + \left(\frac{-1}{2}\sin\left(\frac{t}{2}\right)\right)^2}\, dt}$$