Math 125 H - Winter 2015
Midterm Exam Number Two
February 26, 2015

Name: ______________________ Student ID no. : ______________

Signature: ______________________ Section: ______________

1	14	
2	14	
3	8	
4	6	
5	12	
6	6	
Total	60	

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	–

- The exam consists of SIX problems on SIX pages, including this cover sheet.
- Show all work for full credit.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you write on the back of the page, please indicate that you have done so!
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 80 minutes to complete the exam.

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1. **[7 points per part]** Here are a bunch of integrals. Evaluate them.

(a) $\int 3\sin^4(x)\cos^5(x)\,dx$.

$$= \int 3\sin^4(x)\left(\cos^2(x)\right)^2\cos(x)\,dx$$

$$= \int 3\sin^4(x)\left(1-\sin^2(x)\right)^2\cos(x)\,dx$$

$u = \sin(x)$

$du = \cos(x)\,dx$

$$= \int 3u^4(1-u^2)^2\,du = \int 3u^4(1-2u^2+u^4)\,du = \int \left(3u^4 - 6u^6 + 3u^8\right)du$$

$$= \frac{3}{5}u^5 - \frac{6}{7}u^7 + \frac{3}{9}u^9 + C = \boxed{\frac{3}{5}\sin^5(x) - \frac{6}{7}\sin^7(x) + \frac{3}{9}\sin^9(x) + C}$$

(b) $\int \sin(2x)e^{3x}\,dx$

$u = \sin(2x)$ $\quad v = \frac{1}{3}e^{3x}$

$du = 2\cos(2x)dx$ $\quad dv = e^{3x}dx$

$$= \frac{1}{3}\sin(2x)e^{3x} - \frac{2}{3}\int \cos(2x)e^{3x}\,dx$$

$u = \cos(2x)$ $\quad v = \frac{1}{3}e^{3x}$

$du = -2\sin(2x)\,dx$ $\quad dv = e^{3x}\,dx$

$$\int \sin(2x)e^{3x}dx = \frac{1}{3}\sin(2x)e^{3x} - \frac{2}{9}\cos(2x)e^{3x} - \frac{4}{9}\int \sin(2x)e^{3x}\,dx$$

$$+\frac{4}{9}\int \sin(2x)e^{3x}dx \qquad\qquad +\frac{4}{9}\int \sin(2x)e^{3x}dx$$

$$\boxed{\int \sin(2x)e^{3x}dx = \frac{\frac{1}{3}\sin(2x)e^{3x} - \frac{2}{9}\cos(2x)e^{3x}}{13/9} + C}$$

2. **[7 points per part]** Good news! We haven't run out of integrals yet.

(a) $\int_2^3 \frac{2x^2+9x-3}{x^3-x^2+x-1}\,dx$

factor: $(x-1)(x^2+1)$

$$= \int_2^3 \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+1}\right)dx \qquad \frac{2x^2+9x-3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$2x^2+9x-3 = A(x^2+1) + (Bx+C)(x-1)$$

$x=0$: $-3 = A - C$ $\qquad A = 4$

$x=1$: $8 = 2A$ $\qquad B = -2$

$x=2$: $23 = 5A + 2B + C$ $\qquad C = 7$

$$= \int_2^3 \left(\frac{4}{x-1} + \frac{-2x}{x^2+1} + \frac{7}{x^2+1}\right)dx$$

$u = x^2+1$ ⋮

$$= \left(4\ln|x-1| - \ln|x^2+1| + 7\arctan(x)\right]_2^3 = \boxed{\left(4\ln(2) - \ln(10) + 7\arctan(3)\right) - \left(\underbrace{4\ln(1)}_{0} - \ln(5) + 7\arctan(2)\right)}$$

(b) $\int_{-1}^0 \frac{x}{(-x^2-2x+3)^{5/2}}\,dx$

$$= \int_{-1}^0 \frac{x}{(4-(x+1)^2)^{5/2}}\,dx = \int_0^{\pi/6} \frac{2\sin\theta - 1}{(4-4\sin^2\theta)^{5/2}}\,2\cos\theta\,d\theta$$

$x+1 = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$\left(\theta = \arcsin\left(\frac{x+1}{2}\right)\right)$

$$= \int_0^{\pi/6} \frac{2\sin\theta - 1}{(4\cos^2(\theta))^{5/2}}\,2\cos\theta\,d\theta$$

$$= \int_0^{\pi/6} \frac{2\sin\theta - 1}{16\cos^4\theta}\,d\theta = \int_0^{\pi/6}\left(\frac{1}{8}\tan\theta\sec^3\theta - \frac{1}{16}\sec^4\theta\right)d\theta =$$

$$= \frac{1}{8}\int_0^{\pi/6} \tan\theta\sec^3\theta\,d\theta - \frac{1}{16}\int_0^{\pi/6}\sec^2(\theta)(1+\tan^2\theta)\,d\theta = \frac{1}{8}\left(\frac{\sec^3\theta}{3}\right]_0^{\pi/6} - \frac{1}{16}\left(\tan(\theta) + \frac{\tan^3(\theta)}{3}\right]_0^{\pi/6}$$

$u = \sec\theta$ ⋮ $\qquad u = \tan\theta$ ⋮

$$= \boxed{\frac{1}{8}\left(\frac{\left(\frac{2}{\sqrt{3}}\right)^3}{3} - \frac{1}{3}\right) - \frac{1}{16}\left(\frac{1}{\sqrt{3}} + \frac{\left(\frac{1}{\sqrt{3}}\right)^3}{3}\right)} \quad \left(\text{or } \boxed{\frac{\sqrt{3}}{72} - \frac{1}{24}}\right)$$

3. **[8 points]** Let $f(x)$ be a function such that $f(x) > 0$ on the interval $(0, \infty)$, and $f(x)$ is continuous on the interval $[0, \infty)$.

Let $\mathcal{R}$ be the region in the first quadrant bounded by $y = f(x)$, $x = 0$, and $x = a$.

Let $\mathcal{S}_x$ be the solid formed by revolving $\mathcal{R}$ around the x-axis, and let $\mathcal{S}_y$ be the solid formed by revolving $\mathcal{R}$ around the y-axis.

Find a function $f(x)$ such that for **all** $a > 0$, the solids $\mathcal{S}_x$ and $\mathcal{S}_y$ are equal in volume.

S_x: $\int_0^a \pi (f(x))^2 \, dx$

definitely equal if $\pi (f(x))^2 = 2\pi x f(x)$

S_y: $\int_0^a 2\pi x f(x) \, dx$

$\boxed{f(x) = 2x}$ should work!

4. **[6 points]** Use Simpson's Rule with $n = 6$ to estimate the average value of $f(x) = 2^{(x^2-3)}$ on the interval $[-4, 8]$. You do not need to simplify your answer!

$$\frac{1}{8-(-4)} \int_{-4}^{8} \left(2^{x^2-3}\right) dx$$

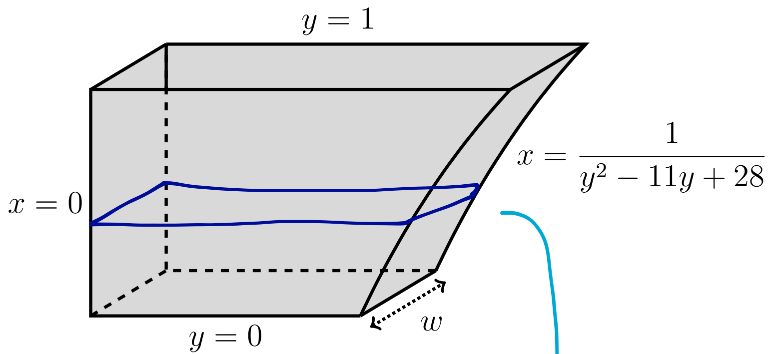
Simpson's Rule: $\Delta x = \frac{12}{6} = 2$

$$\boxed{\frac{1}{12} \frac{2}{3} \left(2^{13} + 4 \cdot 2^{1} + 2 \cdot 2^{-3} + 4 \cdot 2^{1} + 2 \cdot 2^{13} + 4 \cdot 2^{33} + 2^{61} \right)}$$

5. **[12 points]** The front of an aquarium tank is shaped like the region in the first quadrant bounded by $y = 1$ and $x = \dfrac{1}{y^2 - 11y + 28}$.

The aquarium itself is a prism, and the two bases are w meters apart.

The tank is filled with a liquid of density D. Let g be the acceleration due to gravity.



Compute the work needed to empty the tank by pushing all the liquid to the very top.

(Your answer will include w, D, and g.)

Let's look at a thin slice at height y_i.

$$\text{Volume} = \frac{1}{y_i^2 - 11y_i + 28}\, w\,\Delta y$$

So work required to pump out that piece is:

$$\underbrace{\underbrace{\underbrace{\left(\frac{1}{y_i^2-11y_i+28}\right) w\,\Delta y}_{\text{vol}}\, D}_{\text{mass}}\, g}_{\text{force}}\ \underbrace{(1-y_i)}_{\text{dist}}$$

$$\frac{1-y}{(y-4)(y-7)} = \frac{A}{y-4} + \frac{B}{y-7}$$

$1-y = A(y-7) + B(y-4)$

$y=4$: $-3 = -3A$, $A = 1$

$y=7$: $-6 = 3B$, $B = -2$

So in total, want an integral:

$$\int_0^1 wDg\,\frac{1-y}{\underbrace{y^2-11y+28}_{(y-4)(y-7)}}\,dy = wDg\int_0^1\left(\frac{A}{y-4} + \frac{B}{y-7}\right) = wDg\int_0^1\left(\frac{1}{y-4} - \frac{2}{y-7}\right)dy$$

$$= wDg\Big(\ln|y-4| - 2\ln|y-7|\Big)\Big]_0^1 = \boxed{wDg\Big(\ln(3) - 2\ln(6) - \ln(4) + 2\ln(7)\Big)}$$

or $\boxed{wDg\ln\left(\frac{49}{48}\right)}$

6. **[6 points]** Does the integral $\int_0^{\infty} \frac{\sin^2(x)}{x^2 + \sqrt{x}}\, dx$ converge or diverge? Explain.

Hm, it's improper on both sides. But...

$$\int_0^1 \frac{\sin^2(x)}{x^2+\sqrt{x}}\,dx + \int_1^{\infty} \frac{\sin^2(x)}{x^2+\sqrt{x}}\,dx$$

compare:

$$\frac{\sin^2(x)}{x^2+\sqrt{x}} \le \frac{1}{\sqrt{x}}$$

$\int_0^1 \frac{1}{\sqrt{x}}$ converges!

compare:

$$\frac{\sin^2(x)}{x^2+\sqrt{x}} \le \frac{1}{x^2}$$

$\int_1^{\infty} \frac{1}{x^2}\,dx$ converges!

So it $\boxed{\text{converges.}}$

I feel like you probably don't need a whole page for that problem, so here's a Sudoku. Boxes with slashes contain two digits, with the lower number on top.

2/6	3	5	1/4	7/9	8
1/9	4	7/8	3	5	2/6
3/4	7	2/9	8	6	1/5
8	5/6	1	2/9	3/4	7
5	8/9	6	7	1/2	3/4
7	1/2	3/4	5/6	8	9