Math 125 H - Winter 2015
Midterm Exam Number One
January 29, 2015

Name: ______________________ Student ID no. : ______________

Signature: __________________________ Section: __________

1	12	
2	10	
3	7	
4	6	
5	8	
6	5	
7	12	
Total	60	

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	–

- This exam consists of SEVEN problems on SIX pages, including this cover sheet.
- Show all work for full credit.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- If you write on the back of the page, please indicate that you have done so!
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 80 minutes to complete the exam.

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1. **[4 points per part]** Evaluate each integral. You may use any techniques you know.

(a) $\int \left(e^{3x} - \sin^2(x)\cos(x)\right) dx$

$$\frac{1}{3}\int 3e^{3x}\,dx - \int \sin^2(x)\cos x\,dx = \frac{1}{3}\int e^u\,du - \int v^2\,dv$$

$u = 3x$

$du = 3dx$

$v = \sin(x)$

$dv = \cos(x)dx$

$$= \frac{e^u}{3} - \frac{v^3}{3} + C$$

$$\boxed{= \frac{e^{3x}}{3} - \frac{\sin^3(x)}{3} + C}$$

(b) $\frac{1}{2}\int \frac{2\sec^2(\ln(x^2))}{x}\,dx = \frac{1}{2}\int \sec^2(u)\,du = \frac{1}{2}\tan(u) + C$

$u = \ln(x^2)$

$du = \frac{1}{x^2}\cdot 2x\,dx = \frac{2}{x}\,dx$

$$= \boxed{\frac{1}{2}\tan\left(\ln\left(x^2\right)\right) + C}$$

(c) $\frac{1}{2}\int 2x^3\sqrt{x^2+4}\,dx = \frac{1}{2}\int (u-4)\sqrt{u}\,du = \frac{1}{2}\int \left(u^{3/2} - 4u^{1/2}\right)du = \frac{u^{5/2}}{5} - \frac{4u^{3/2}}{3} + C$

$u = x^2 + 4$

$du = 2x\,dx$

$$= \boxed{\frac{(x^2+4)^{5/2}}{5} - \frac{4(x^2+4)^{3/2}}{3} + C}$$

2. **[10 points]** A particle is moving along the x-axis.

At time $t \geq 0$ seconds, its acceleration is given by $a(t) = 2t - 8$.

At time $t = 0$ it's at $x = 3$, and at time $t = 3$ it's at $x = 21$.

In the first 8 seconds, what is the **total distance traveled** by the particle?

$a(t) = 2t - 8$

$v(t) = t^2 - 8t + C_1$

$s(t) = \frac{1}{3}t^3 - 4t^2 + C_1 t + C_2$

$s(0) = C_2 = 3$

$s(3) = \frac{1}{3}(3)^3 - 4(3)^2 + C_1(3) + 3 = 21$

$9 - 36 + 3C_1 + 3 = 21$

$3C_1 = 45$

$C_1 = 15$

$s(t) = \frac{1}{3}t^3 - 4t^2 + 15t + 3$

For total distance need to know when it turns around:

$v(t) = t^2 - 8t + 15 = 0$

$(t-3)(t-5) = 0$

So: Where is the particle at $t = 0, 3, 5, 8$?

$\left.\begin{array}{l} s(0) = 3 \\ s(3) = 21 \end{array}\right\}$ given

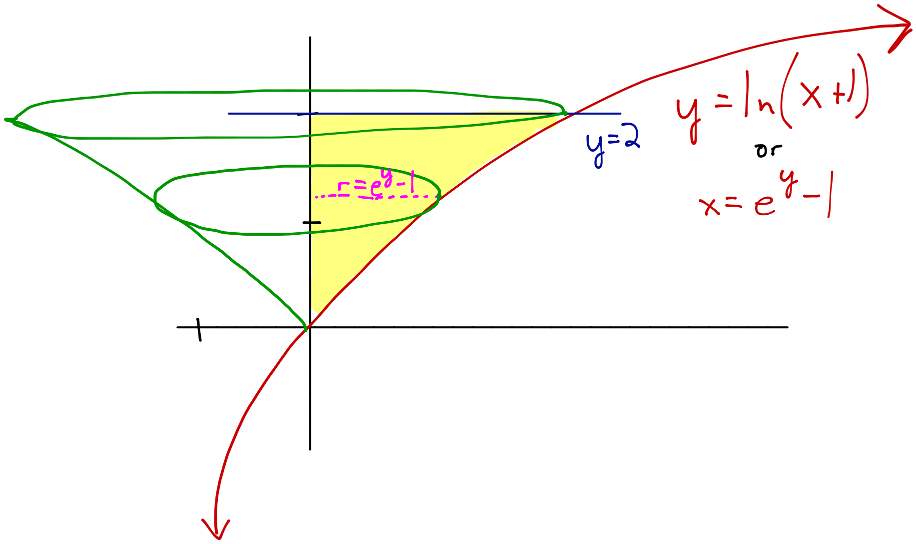
$s(5) = \frac{1}{3}(5)^3 - 4(5)^2 + 15(5) + 3 = \frac{59}{3}$

$s(8) = \frac{1}{3}(8)^3 - 4(8)^2 + 15(8) + 3 = \frac{113}{3}$

Total distance $= (21 - 3) + \left(21 - \frac{59}{3}\right) + \left(\frac{113}{3} - \frac{59}{3}\right)$

$= 18 + \frac{4}{3} + 18 = \boxed{\frac{112}{3} \text{ units}}$

3. **[7 points]** Let $\mathcal{R}$ be the region bounded by $y = \ln(x+1)$, the y-axis, and the line $y = 2$. Compute the volume of the solid obtained by revolving $\mathcal{R}$ around the y-axis.



Integrate w.r.t. y:

$$\int_0^2 \pi\left(e^y - 1\right)^2 dy = \pi \int_0^2 \left(e^{2y} - 2e^y + 1\right) dy$$

$$= \pi\left(\frac{1}{2}e^{2y} - 2e^y + y\right)\Big]_0^2$$

$$= \pi\left(\left(\frac{1}{2}e^4 - 2e^2 + 2\right) - \left(\frac{1}{2}e^0 - 2e^0 + 0\right)\right)$$

$$= \boxed{\pi\left(\frac{e^4}{2} - 2e^2 - \frac{1}{2}\right)}$$

4. **[6 points]** Use any techniques you'd like to compute the following limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\frac{i\pi}{3n}\right)\left(\frac{\pi}{12n}\right)$$

$4i\Delta x$
$= 4x_i$, so $a = 0$,
$b = \frac{\pi}{12}$

Δx, so $b - a = \frac{\pi}{12}$

$$= \int_0^{\frac{\pi}{12}} \tan(4x)\,dx = \frac{1}{4}\int_0^{\frac{\pi}{12}} \frac{4\sin(4x)}{\cos(4x)}\,dx$$

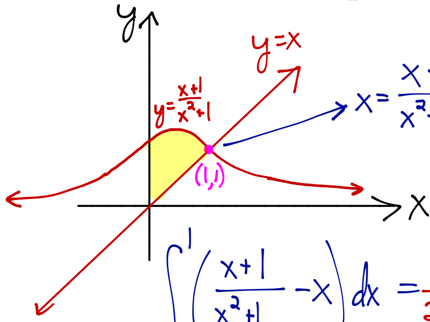
$u = \cos(4x)$, $du = -4\sin(x)dx$

$$= \frac{-1}{4}\int_1^{1/2} \frac{1}{u}\,du$$

$$= \frac{-1}{4}\left(\ln|u|\right)\Big]_1^{1/2} = \frac{-1}{4}\left(\ln\left(\tfrac{1}{2}\right) - \ln(1)\right)$$

$$= \boxed{\frac{\ln(2)}{4}}$$

5. **[8 points]** Compute the area of the region bounded by $y = x$, $y = \dfrac{x+1}{x^2+1}$, and the y-axis.



$x = \dfrac{x+1}{x^2+1} \rightarrow x^3 + x = x + 1 \rightarrow x = 1$

$$\int_0^1 \left(\frac{x+1}{x^2+1} - x\right) dx = \frac{1}{2}\int_0^1 \underbrace{\frac{2x}{x^2+1}}_{} dx + \int_0^1 \frac{1}{x^2+1} dx - \int_0^1 x\, dx$$

$u = x^2 + 1$

$du = 2x\,dx$

$$= \frac{1}{2}\int_1^2 \frac{du}{u} + \arctan(x)\Big]_0^1 - \frac{x^2}{2}\Big]_0^1 = \frac{1}{2}\Big(\ln|u|\Big)\Big]_1^2 + \arctan(1) - \frac{1}{2}$$

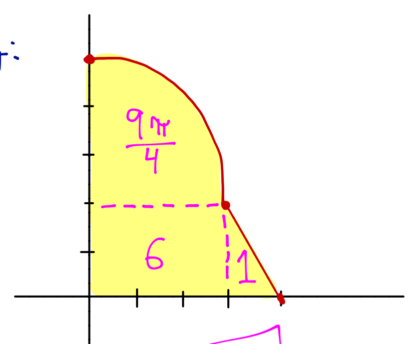
$$= \boxed{\frac{1}{2}\ln(2) + \frac{\pi}{4} - \frac{1}{2}}$$

6. **[5 points]** Consider the following function $f(x)$:

$$f(x) = \begin{cases} 2 + \sqrt{9 - x^2} & \text{if } 0 \leq x \leq 3 \\ 8 - 2x & \text{if } 3 < x \leq 4 \end{cases}$$

Compute $\displaystyle\int_0^4 f(x)\, dx$.

Draw it:

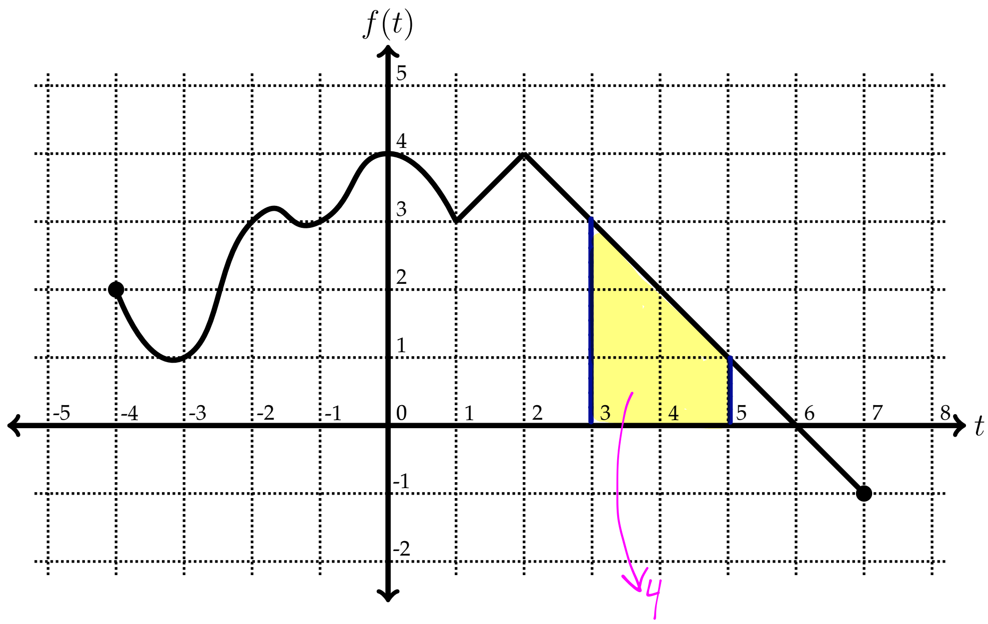


$$\boxed{\text{Area} = 7 + \frac{9\pi}{4}}$$

7. **[4 points per part]** You awake to a large commotion outside your window.

"It's the graph question," cries a youth. "Jonah wrote another graph question!"

A church bell chimes, and a parade makes its way through the plaza. This is a good day.



(a) Let $h(x) = \int_{e^x}^{x} \cos(\pi t) f(t)\, dt$. Compute $h'(0)$.

Let $g(x) = \int_0^x \cos(\pi t) f(t)\, dt$.

$g'(x) = \cos(\pi x) f(x)$.

$h(x) = g(x) - g(e^x)$, so $h'(x) = g'(x) - g'(e^x)e^x = \cos(\pi x) f(x) - \cos(\pi e^x) f(e^x) e^x$

$h'(0) = \cos(0) f(0) - \cos(\pi e^0) f(e^0) e^0$

$= f(0) + f(1) = 4 + 3 = \boxed{7}$

(b) Compute $\int_1^2 2^t f(2^t + 1)\, dt$.

$u = 2^t + 1$

$du = 2^t \ln(2)\, dt$

$\frac{1}{\ln(2)} \int_1^2 2^t \ln(2) f(2^t + 1)\, dt = \frac{1}{\ln(2)} \int_3^5 f(u)\, du = \boxed{\frac{4}{\ln(2)}}$

(c) Suppose the area under $f(t)$ from -4 to 2 is revolved around the horizontal axis.

Write (but don't evaluate) an integral for the resulting volume, and compute the M_3 approximation of that integral.

$\int_{-4}^{2} \pi (f(t))^2\, dt$

Δx

$M_3 = 2 \cdot \pi \left(f(-3)^2 + f(-1)^2 + f(1)^2 \right) = 2 \cdot \pi (1^2 + 3^2 + 3^2) = \boxed{38\pi}$