DFEP #1: Wednesday, January 13th.

Write each limit as an integral, then compute it.

(a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{3n}\right)\frac{\pi}{3n}$

(c) $\displaystyle\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-i^2}}$

DFEP #1 Solution:

(a) We want to write $\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{3n}\right)\frac{\pi}{3n}$ as an integral. Recall that using a limit of right-hand Riemann sums, we have $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$, where $\Delta x = \frac{b-a}{n}$.

Stare at this limit a bit and you can see it's what you get when $a = 0$, $b = \frac{\pi}{3}$, and $f(x) = \cos(x)$.

So it's

$$\int_0^{\pi/3}\cos(x)\,dx = \sin(x)\Big]_0^{\pi/3} = \sin\left(\frac{\pi}{3}\right) - \sin(0) = \frac{\sqrt{3}}{2}.$$

(b) Hey, I warned you, this one is tough. First, a little algebra:

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{n^2-i^2}} = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{n\sqrt{1-\left(\frac{i}{n}\right)^2}} = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{1}{\sqrt{1-\left(\frac{i}{n}\right)^2}}\cdot\frac{1}{n}$$

This is a limit of *left*-hand Riemann sums equal to $\int_0^1 \frac{1}{\sqrt{1-x^2}}$. The antiderivative here is straightforward enough and we arrive at

$$\arcsin(1) - \arcsin(0) = \frac{\pi}{2}.$$

"But wait!" I hear you cry. "The integrand isn't continuous at $x = 1$!" Good point, good point. It turns out this works out okay since the antiderivative is defined at $x = 1$, and there are no other discontinuities. But the exact reasons for why this is allowed will come in a later chapter.

DFEP #2: Friday, January 15th.

Consider the function $f(x) = \int_{\sin(2x)}^{3x} e^{t^2}\,dt$.

Is $f(x)$ concave up or concave down at $x = \frac{\pi}{2}$?

DFEP #2 Solution:

Okay, we need the second derivative of $f(x) = \displaystyle\int_{\sin(2x)}^{3x} e^{t^2}\,dt$.

Let $g(x) = \int_0^x e^{t^2}\,dt$. According to the fundamental theorem of calculus, $g'(x) = e^{x^2}$. Furthermore, $f(x) = g(3x) - g(\sin(2x))$. So:

$$f'(x) = 3g'(3x) - 2\cos(2x)g'(\sin(2x)) = 3e^{(3x)^2} - 2\cos(2x)e^{\sin^2(2x)}$$

Differentiating again gives:

$$f''(x) = 54xe^{(3x)^2} + 4\sin(2x)e^{\sin^2(2x)} - 8\cos^2(2x)\sin(2x)e^{\sin^2(2x)}$$

And therefore $f''\left(\frac{\pi}{2}\right) = 54\left(\frac{\pi}{2}\right)e^{(3\pi/2)^2} > 0$, so the function is concave up.

DFEP #3: Wednesday, January 20th.

Your train leaves New York for Philadelphia at 9:00 AM at a speed of 100 miles per hour. Seated next to you is a man staring at a page of tricky integrals. Solve the integrals for him.

(a) $\displaystyle\int \frac{1}{x\ln(x^2)}\,dx$

(b) $\displaystyle\int e^{e^x + x}\,dx$

(c) $\displaystyle\int_0^3 x^5\sqrt[3]{x^2 - 16}\,dx$

DFEP #3 Solution:

(a) Set $u = \ln(x^2)$, $du = \dfrac{2x}{x^2}\,dx = \dfrac{2}{x}\,dx$.

So $\displaystyle\int \frac{1}{x\ln(x^2)}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|\ln(x^2)| + C.$

(b) Rewrite as $\displaystyle\int e^x e^{e^x}\,dx$. Then $u = e^x$, $du = e^x\,dx$ gives:

$\displaystyle\int e^u\,du = e^u + C = e^{e^x} + C.$

(c) Let $u = x^2 - 16$, so $x^4 = (u+16)^2$ so the integral becomes

$$\frac{1}{2}\int_{-16}^{-7} (u+16)^2 \sqrt[3]{u}\,du$$

Expand to get

$$\frac{1}{2}\int_{-16}^{-7} \left(u^{7/3} + 32u^{4/3} + 256u^{1/3}\right)\,du$$

which is solved easily enough by the power rule:

$$\frac{1}{2}\left[\frac{3u^{10/3}}{10} + \frac{96u^{7/3}}{7} + 192u^{4/3}\right]_{-16}^{-7} \approx -254.1$$

DFEP #4: Friday, January 22nd.

Find a such that the area of the region in the first quadrant bounded by $y = x^3$ and $y = ax$ is 5.

DFEP #4 Solution:

What are the bounds on this region? $x^3 = ax$ means that $x = 0$, $x = \sqrt{a}$, or $x = -\sqrt{a}$. The region in the first quadrant runs from 0 to $\sqrt{a}$, and over that interval, $ax \geq x^3$

So the area in question is

$$\int_0^{\sqrt{a}} (ax - x^3)\, dx = \frac{ax^2}{2} - \frac{x^4}{4}\Big]_0^{\sqrt{a}} = \frac{a^2}{4} = 5$$

So $a = \sqrt{20}$.

DFEP #5: Monday, January 25th.

Compute the area of the solid formed by rotating the region between $y = e^x$ and $y = \dfrac{x-1}{x}$ from $x = 1$ to $x = 4$ around the line $y = -1$.

DFEP #5 Solution:

On the interval $[1,4]$, we know that $\dfrac{x-1}{x} < 1 < e \le e^x$, so $y = \dfrac{x-1}{x}$ is always below $y = e^x$. That means the volume is given by:

$$\int_1^4 \pi\left((e^x+1)^2 - \left(\frac{x-1}{x}+1\right)^2\right)\,dx$$

This isn't too bad. (To integrate e^{2x}, just use $u = 2x$.)

$$\begin{aligned}
&\pi\int_1^4 \left(e^{2x} + 2e^x + 1 - \frac{4x^2-4x+1}{x^2}\right)\,dx \\
&= \pi\int_1^4 \left(e^{2x} + 2e^x + 1 - 4 - \frac{4}{x} + \frac{1}{x^2}\right)\,dx \\
&= \pi\left(\frac{1}{2}e^{2x} + 2e^x - 3x - 4\ln|x| - \frac{1}{x}\right)\Big]_1^4 \\
&= \pi\left(\frac{1}{2}e^8 + 2e^4 - 12 - 4\ln|4| - \frac{1}{4}\right) - \pi\left(\frac{1}{2}e^2 + 2e - 3 - 0 - \frac{1}{1}\right)
\end{aligned}$$

DFEP #6: Friday, January 29th.

A hemispherical pot with diameter 50 cm is filled to the brim with tomato soup of uniform density 1500 kg/m^3. Find the work required to drink all of the soup with a straw. (The top of the straw is level with the rim of the tank.)

DFEP #6 Solution:

Let's imagine cutting the soup into horizontal cross sections (it's a very thick soup, I guess): how much work does it take to bring each cross section to the top?

Say $y = 0$ is the bottom of the pot and $y = 0.5$ is the top of the pot. (We're working in meters.) At height y, we get a slice with area $\pi\left(\sqrt{0.5^2-(0.5-y)^2}\right)^2 = \pi(y-y^2)$, so the mass at that point is $\pi(y-y^2)(1500)\,dy$, and the work required is $9.8 \cdot 1500\pi(y-y^2)(0.5-y)\,dy$.

So the total work required is

$$\begin{aligned}
&\int_0^{.5} 9.8 \cdot 1500\pi(y^3 - 1.5y^2 + 0.5y)\,dy \\
&= 9.8 \cdot 1500\pi\left(0.25y^4 - 0.5y^3 + 0.25y^2\right)\Big]_0^{0.5} \\
&= 9.8 \cdot 1500\pi\left(\frac{1}{64} - \frac{1}{16} + \frac{1}{16}\right) \\
&= \frac{9.8 \cdot 1500\pi}{64}\ \mathrm{J}
\end{aligned}$$

(Note: a previous version of this solution was missing a factor of 9.8.)

DFEP #7: Monday, February 1st.

Determine the average value of the function $f(x) = e^{2x}(x^2 - 5x + 3)$ on the interval $[0, 2]$.

DFEP #7 Solution:

Okay, we want the average value of $f(x) = e^{2x}(x^2 - 5x + 3)$ on the interval $[0, 2]$, so we want to find $\frac{1}{2}\int_0^2 e^{2x}(x^2 - 5x + 3)\,dx$. Let's use integration by parts! $u = x^2 - 5x + 3$, $dv = e^{2x}\,dx$, so $du = (2x - 5)\,dx$, and $v = \frac{1}{2}e^{2x}$. So:

$$\frac{1}{2}\int_0^2 e^{2x}(x^2 - 5x + 3)\,dx = \frac{1}{2}\left(\frac{1}{2}e^{2x}(x^2 - 5x + 3)\Big]_0^2 - \frac{1}{2}\int_0^2 e^{2x}(2x - 5)\,dx\right)$$

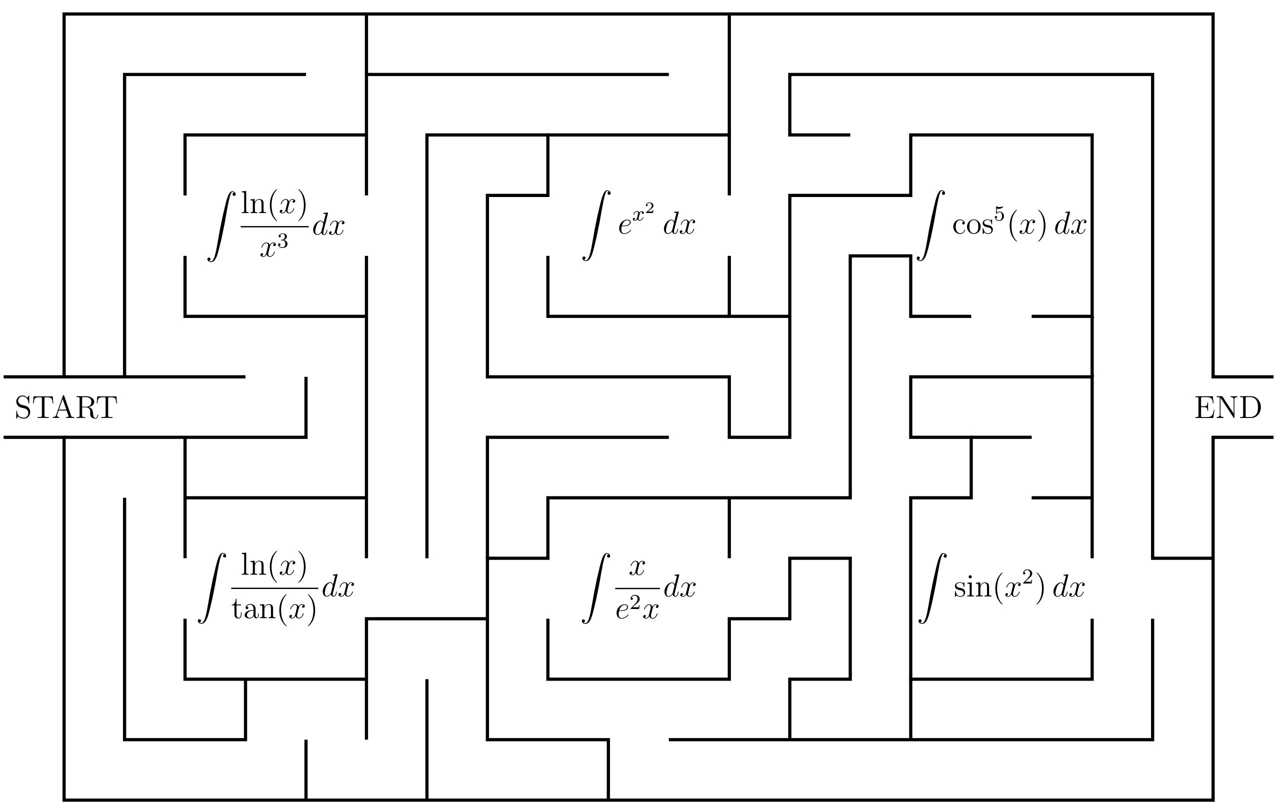
Again! $u = (2x - 5)$, $dv = e^{2x}$, etc:

$$= \frac{1}{2}\left(\left(\frac{1}{2}(e^{2x}(x^2 - 5x + 3)) - \frac{1}{4}e^{2x}(2x - 5)\right)\Bigg]_0^2 + \frac{1}{2}\int_0^2 e^{2x}\,dx\right)$$

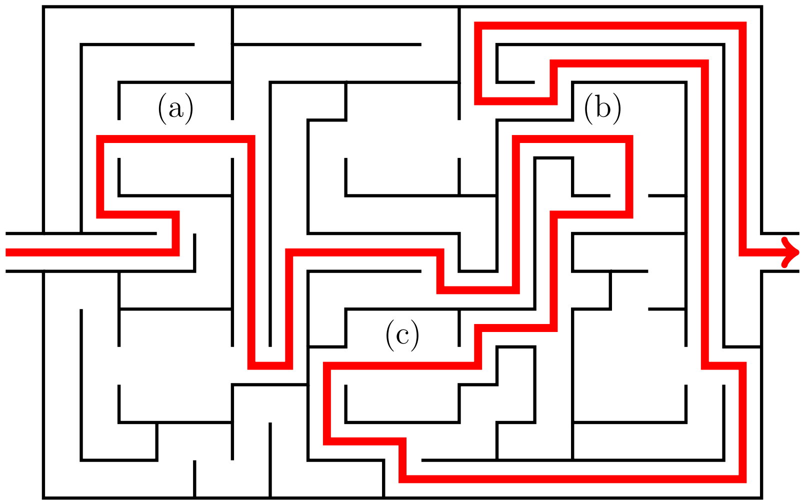
This is fun, right? Anyway you get $-3 - e^4$.

DFEP #8: Wednesday, February 3rd.

Complete the following maze. Evaluate all integrals along your path.



DFEP #8 Solution:



(a) $\displaystyle\int \frac{\ln(x)}{x^3}dx$. Let $u = \ln(x)$, $dv = x^{-3}dx$, then this becomes $\displaystyle -\frac{\ln(x)}{2x^2} + \int \frac{1}{2x^3}dx$, or $\displaystyle -\frac{\ln(x)}{2x^2} - \frac{1}{4x^2} + C$.

(b) $\displaystyle\int \cos^5(x)\,dx = \int \cos(x)(1-\sin^2(x))^2\,dx = \int (1-u^2)^2\,du = \int (1-2u^2+u^4)\,du$. Integrate and resubstitute to get $\displaystyle \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C$.

(c) $\displaystyle\int \frac{x}{e^{2x}}dx$. Let $u = x$, $dv = e^{-2x}dx$, so $du = dx$, $v = \dfrac{-1}{2}e^{-2x}\,dx$ and we get $\displaystyle \frac{-x}{2e^{2x}} + \int \frac{e^{-2x}}{2}\,dx = \frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}} + C$.

DFEP #9: Friday, February 6th.

Let $\mathcal{R}$ be the region bounded by $y = 0$, $x = 0$, $x = 4$, and $y = \sqrt{x^2+6x+25}$. Compute the volume of the solid formed by revolving $\mathcal{R}$ around the y-axis.

DFEP #9 Solution:

Using the shell method, we want $\int_0^4 2\pi x\sqrt{x^2+6x+25}\,dx$. Let's complete the square to get

$$2\pi\int_0^4 x\sqrt{(x+3)^2+16}\,dx$$

Setting $u = x+3$, gives $2\pi\int_3^7 (u-3)\sqrt{u^2+16}\,du$, and now we can use trigonometric substitution with $u = 4\tan(\theta)$, $du = 4\sec^2(\theta)d\theta$, leaving the integral

$$2\pi\int_{\arctan(3/4)}^{\arctan(7/4)} (4\tan(\theta)-3)\sqrt{4\sec^2\theta}4\sec^2\theta\,d\theta$$

This simplifies to:

$$\begin{aligned}
&16\pi\int_{\arctan(3/4)}^{\arctan(7/4)} (4\tan(\theta)\sec^3(\theta) - 3\sec^3(\theta))\,d\theta \\
&= 16\pi\left(4\sec^3(\theta)/3 - \frac{3}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|\right)\right)\Big]_{\arctan(3/4)}^{\arctan(7/4)}
\end{aligned}$$

I can't really blame you if you don't want to write all that out.

DFEP #10: Monday, February 8th.

Compute the area of the region bounded by the curves $y = 2x$, $y = \dfrac{2x^3+10x^2+16x+7}{x^2+5x+6}$, $x = 0$, and $x = 2$.

DFEP #10 Solution:

We want $\displaystyle\int_0^2 \left(\frac{2x^3+10x^2+16x+7}{x^2+5x+6} - 2x\right) dx = \int_0^2 \frac{4x+7}{(x+1)(x+5)}\, dx$. Okay, cool, let's do partial fractions:

$$\begin{aligned}
&\int_0^2 \left(\frac{3/4}{x+1} + \frac{13/4}{x+5}\right) dx \\
&= \left(\frac{3}{4}\ln|x+1| + \frac{13}{4}\ln|x+5|\right)\Bigg]_0^2 \\
&= \frac{3}{4}\ln(3) + \frac{13}{4}\ln(7) - \frac{13}{4}\ln(6)
\end{aligned}$$

DFEP #11: Wednesday, February 10th.

Jeannette is three years older than Hortense, and in two years Hortense will be twice as old as Bertrand. Bertrand is taller than Hortense, but shorter than Jeanette.

Anyway, here are some integrals.

(a) $\displaystyle\int \frac{4x^4+10x^3+9x^2+16x+8}{x^3+2x^2+x} dx$

(b) $\displaystyle\int \sin^5(x)\cos^8(x)\, dx$

(c) $\displaystyle\int \frac{\sqrt{16+x^2}}{x} dx$

DFEP #11 Solution:

Jeanette is shorter than Brutus, but taller than Imhotep. Also, integrals:

(a) Long division and partial fractions yields

$$\int \left(\frac{5}{(x+1)^2} - \frac{7}{x+1} + \frac{8}{x} + 4x + 2 \right) dx$$
$$= \frac{-5}{x+1} - 7\ln|x+1| + 8\ln|x| + 2x^2 + 2x + C$$

(b) First, rewrite as:

$$\int \sin(x) \left(\sin^2(x)\right)^2 \cos^8(x)\, dx = \int \sin(x) \left(1-\cos^2(x)\right)^2 \cos^8(x)\, dx$$

Then use $u = \cos(x)$, $du = -\sin(x)\, dx$ to get:

$$\int -(1-u^2)^2 u^8\, du = \int (-u^{12} + 2u^{10} - u^8)\, du = -\frac{u^{13}}{13} + \frac{2u^{11}}{11} - \frac{u^9}{9} + C$$

and resubstitute:

$$-\frac{\cos^{13}(x)}{13} + \frac{2\cos^{11}(x)}{11} - \frac{\cos^9(x)}{9} + C$$

(c) We want $\int \frac{\sqrt{16+x^2}}{x} dx$. Looks like a job for trigonometric substitution, no?

Let $x = 4\tan(\theta)$ and the integral simplifies to $\int \frac{4\sec^3(\theta)}{\tan(\theta)} d\theta$. Use the identity $\sec^2(\theta) = 1 + \tan^2(\theta)$ and this simplifies to

$$\int (\csc(\theta) + \sec(\theta)\tan(\theta))\, d\theta = 4\left(\ln|\csc(\theta) - \cot(\theta)| + \sec(\theta)\right) + C$$

Finally, using a comparison triangle to resubstitute:

$$4\left(\ln\left| \frac{\sqrt{x^2+16}}{x} - \frac{4}{x} \right| + \frac{\sqrt{x^2+16}}{4} \right) + C$$

DFEP #12: Friday, February 12th.

Use Simpson's Rule with $n = 6$ to estimate the average value of $f(x) = \sqrt{x^3+5}$ on the interval $[-1, 11]$.

DFEP #12 Solution:

So we want to estimate $\frac{1}{12}\int_{-1}^{11}\sqrt{x^3+5}\,dx$ using Simpson's Rule with $n=6$. That means $\Delta x = 2$, so we have:

$$\frac{1}{12}\cdot\frac{2}{3}\left(\sqrt{4}+4\sqrt{6}+2\sqrt{32}+4\sqrt{130}+2\sqrt{348}+4\sqrt{734}+\sqrt{1336}\right)$$

DFEP #13: Wednesday, February 17th.

Determine whether the following integral is convergent or divergent. (You do not need to evaluate the integral.)

$$\int_0^\infty \frac{dx}{\sqrt{x^3+5}}$$

DFEP #13 Solution:

Well, first off, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^3+5}}$ converges because it's just a regular definite integral. What about the rest? On the interval $[1,\infty)$, we know that $\sqrt{x^3+5} > \sqrt{x^3}$, so

$$\frac{1}{\sqrt{x^3+5}} < \frac{1}{x^{3/2}}.$$

Since $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$ converges by the p-test, so does this.

DFEP #14: Friday, February 19th.

Compute $\displaystyle\int_2^5 \left(\ln(x-2) + \frac{1}{\sqrt{5-x}}\right) dx.$

DFEP #14 Solution:

$\displaystyle\int_2^5 \left(\ln(x-2) + \frac{1}{\sqrt{5-x}}\right)\,dx$ is an improper integral for two reasons: $\ln(x-2)$ has an asymptote at 2, and $\dfrac{1}{\sqrt{5-x}}$ has an asymptote at 5. So let's break it into two pieces:

$$\int_2^5 \left(\ln(x-2) + \frac{1}{\sqrt{5-x}}\right)\,dx = \lim_{t\to 2^+}\left(\int_t^5 \ln(x-2)\,dx\right) + \lim_{t\to 5^-}\left(\int_2^t \frac{1}{\sqrt{5-x}}\,dx\right)$$

For the first integral, we use integration by parts and end up with:

$$\lim_{t\to 2^+}\big((x-2)(\ln(x-2)-1)\big)\Big]_t^5 = 3\ln(3) - 3 - \lim_{t\to 2^+}\big((t-2)(\ln(t-2)-1)\big)$$

l'Hôpital's rule tells us that the limit is zero, so this part is just $3\ln(3)-3$. The other part is:

$$\lim_{t\to 5^-}\left(-2\sqrt{5-x}\right)\Big]_2^t = 2\sqrt{3}$$

So in total, the integral is $3\ln(3) - 3 + 2\sqrt{3}$.

DFEP #15: Monday, February 22nd.

Evaluate each integral:

(a) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

(b) $\displaystyle\int \frac{x^5 - 3x^3 + 20x^2 + 11x - 9}{x^3 + 2x^2 - 3x}\,dx$

(c) $\displaystyle\int \sin(\sqrt[3]{x})\,dx$

DFEP #15 Solution:

(a) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

Let's start by substituting $x = \tan(\theta)$, $dx = \sec^2(\theta)d\theta$. That gives us:

$$\int \frac{\sec^2(\theta)}{\tan^2(\theta)\sec(\theta)}d\theta = \int \frac{\cos(\theta)}{\sin^2(\theta)} = \int \frac{du}{u^2} = \frac{-1}{u} + C$$

Resubstitute to get $\dfrac{-1}{\sin(\theta)} + C$. But $\tan(\theta) = x$, so $\sin(\theta) = \dfrac{x}{\sqrt{x^2+1}}$, so we have

$$\int \frac{dx}{x^2\sqrt{x^2+1}} = \frac{-\sqrt{x^2+1}}{x} + C$$

(b) $\displaystyle\int \frac{x^5 - 3x^3 + 20x^2 + 11x - 9}{x^3 + 2x^2 - 3x}\,dx$

Polynomial long division leaves us with $\int \left(x^2 - 2x + 4 + \frac{6x^2+23x-9}{x(x-1)(x+3)}\right)\,dx$, which partial fractions turns into

$$\int \left(x^2 - 2x + 4 + \frac{3}{x} + \frac{5}{x-1} - \frac{2}{x+3}\right)\,dx$$

and, hey, that's easy:

$$\frac{1}{3}x^3 - x^2 + 4x + 3\ln|x| + 5\ln|x-1| - 2\ln|x+3| + C$$

(c) $\displaystyle\int \sin(\sqrt[3]{x})\,dx$

Use $u = \sqrt[3]{x}$, so $du = \frac{1}{3\sqrt[3]{x^2}}dx$, or $dx = 3u^2$. So this becomes

$$\int 3u^2\sin(u)\,du$$

Integrate by parts, twice, to get:

$$-3u^2\cos(u) + 6u\sin(u) + 6\cos(u) + C$$

and resubstitute for

$$-3\sqrt[3]{x^2}\cos(\sqrt[3]{x}) + 6\sqrt[3]{x}\sin(\sqrt[3]{x}) + 6\cos(\sqrt[3]{x}) + C$$

DFEP #16: Friday, February 26th.

Compute the centroid of the region bounded by the curves $y = e^x$, $y = \sin(x)$, $x = 0$, and $x = \pi$.

DFEP #16 Solution:

We want the centroid of the region bounded by $y = e^x$, $y = \sin(x)$, $x = 0$, and $x = \pi$, and to do that we'll need three things: the total mass m, and the moments M_y and M_x. Note that $e^x > \sin(x)$ for this whole region.

$$m = \int_0^\pi \rho(e^x - \sin(x))\, dx = \rho\left(e^x + \cos(x)\right)\Big]_0^\pi = \rho(e^\pi - 3)$$

$$M_y = \int_0^\pi \rho x(e^x - \sin(x))\, dx = \rho x(e^x + \cos(x))\Big]_0^\pi - \int_0^\pi \rho(e^x + \cos(x))\, dx$$

$$= \rho\left(xe^x + x\cos(x) - e^x - \sin(x)\right)\Big]_0^\pi = \rho(\pi e^\pi - \pi - e^\pi + 1)$$

$$M_x = \int_0^\pi \frac{1}{2}\rho\left(e^{2x} - \sin^2(x)\right)\, dx = \rho\left(\frac{1}{4}e^{2x} - \frac{1}{4}x + \frac{1}{8}\sin(2x)\right)\Big]_0^\pi$$

$$M_x = \rho(\frac{1}{4}e^{2\pi} - \frac{\pi}{4} - \frac{1}{4})$$

So the centroid is at $\left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\pi e^\pi - \pi - e^\pi + 1}{e^\pi - 3}, \frac{e^{2\pi} - \pi - 1}{4e^\pi - 12}\right)$.

DFEP #17: Monday, February 29th:

Solve the differential equation $y' = xy\sin(x)$ with initial condition $y(\pi) = 1$.

DFEP #17 Solution:

We want to solve $\dfrac{dy}{dx} = xy\sin(x)$ with $y(\pi) = 1$. Separate variables to get

$$\frac{dy}{y} = x\sin(x)\,dx.$$

Integrate (using integration by parts on the right side) to get

$$\ln|y| = -x\cos(x) + \sin(x) + C$$

Plug in $x = \pi$, $y = 1$ to get $0 = \pi + C$, so $C = -\pi$, and we have the equation

$$\ln|y| = -x\cos(x) + \sin(x) - \pi$$

We want the continuous piece of this curve containing $(\pi, 1)$, so $y > 0$ and we have

$$y = e^{-x\cos(x)+\sin(x)-\pi}.$$

DFEP #18: Wednesday, March 2nd.

Westley has a jug that contains 4 liters of wine with 70 grams of Iocaine powder mixed in. At time $t = 0$, he begins pouring in more wine and Iocaine powder at a rate of 0.5 liters of wine and 2 grams of powder per minute. At the same time, the jug is mixed well and 0.5 liters of the mixture are poured out per minute (and safely disposed of).

1 gram of Iocaine powder is fatal to the average person, but Westley can withstand twice that amount. How long should Westley keep mixing the wine in this way so that he can safely drink a 200 mL glass, but no one else can?

Express your answer as an interval of time, e.g.: “He should stop between 5 and 7 minutes”.

DFEP #18 Solution:

Let $I(t)$ be the amount of Iocaine powder in the jug of wine after t minutes. We know that $I'(t) = (\text{rate in}) - (\text{rate out})$. The rate in is simply 2 grams per minute. The rate out? Well, we're pouring out 0.5 liters (1/8 of the jug) per minute, so that's $I/8$. So we want to solve the differential equation:

$$\frac{dI}{dt} = 2 - \frac{I}{8} \qquad I(0) = 70$$

We can separate the differential equation to get

$$\frac{8\,dI}{16 - I} = dt$$

and integrate to get

$$\int \frac{8\,dI}{16 - I} = \int dt$$

$$-8 \ln|16 - I| = t + C$$

$$I = 16 + Ae^{\frac{-t}{8}}$$

Plugging in $t = 0$, $I = 70$ tells us that $A = 54$. So now we want to know when the wine is safe for Westley to drink, and when it's safe for everyone else. It's safe for Westley when there are 2 grams per 0.2 L, which means there are 40 grams in the 4 liter jug. Likewise, it's safe for everyone else when there are less than 0.1 grams per 0.2 L, which means there are 20 grams in the 4 liter jug.

Plugging in $I = 40$ and $I = 20$ and solving for t yields $t = -8 \ln\left(\frac{24}{54}\right) = 6.49$ minutes when it becomes safe for him to drink, and $t = -8 \ln\left(\frac{4}{54}\right) = 20.82$ minutes when it becomes safe for everyone else.

So he should let this mixing process happen for any time between 6.49 and 20.82 minutes.