An Overview of Vector Algebra

Notation: We can write vectors in a few different ways.

In n dimensions, we can write its components as an ordered n-tuple inside angled brackets. In other words, the vector from the origin to $(a_1, a_2, \ldots, a_n)$ can be written as $\langle a_1, a_2, \ldots, a_n \rangle$.

In two dimensions, we use $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ to represent the standard unit vectors. Similarly, in three dimensions we have:

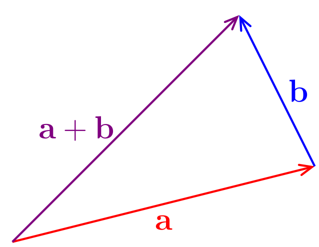
$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

That gives us a second way to write vectors in two or three dimensions, as sums of standard unit vectors: $\langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. For instance, $\langle 2, 0, -7 \rangle = 2\mathbf{i} - 7\mathbf{k}$.

In general, we use **bold** letters to represent vectors in typed documents. When writing by hand, use an overarrow: $\vec{a}, \vec{b}, \vec{c}$ instead of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

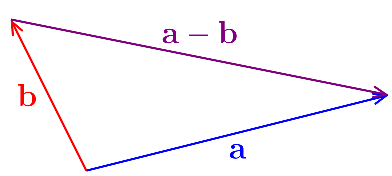
Sums: To add two vectors, simply add their corresponding components: if $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$, then $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \rangle$.

Geometrically, $\mathbf{a} + \mathbf{b}$ can be found by drawing $\mathbf{a}$ and $\mathbf{b}$ tip-to-tail in the plane:



Differences: Likewise, we can subtract one vector from another by taking the differences of their corresponding components: if $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$, then $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n \rangle$.

A common geometric interpretation is given by drawing $\mathbf{a}$ and $\mathbf{b}$ with their tails at the same point, so that $\mathbf{a} - \mathbf{b}$ is the vector from the tip of $\mathbf{b}$ to the tip of $\mathbf{a}$:



Magnitude: The *magnitude* of a vector is its length. The magnitude of an n-dimensional vector $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ is written as $|\mathbf{a}|$ or $||\mathbf{a}||$ and is given by the formula:

$$||\mathbf{a}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Scalar multiplication: A *scalar*, in contrast with a vector, is any real number. We can multiply a vector $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$ by a scalar c to get $c\mathbf{a} = \langle ca_1, \ldots, ca_n \rangle$.

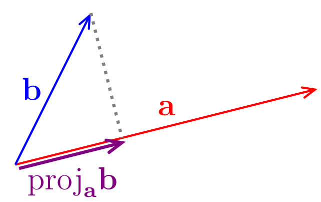
Geometrically, this has the effect of stretching $\mathbf{a}$ by a factor of c. If c is negative, then $c\mathbf{a}$ points in the opposite direction of $\mathbf{a}$. In either case, we say that $\mathbf{a}$ and $c\mathbf{a}$ are *parallel* vectors.

Dot product: Given two vectors $\mathbf{a} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, b_2, \ldots, b_n \rangle$, we can compute the dot product $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$. Notice that the result is a scalar, not a vector.

If the angle between $\mathbf{a}$ and $\mathbf{b}$ (when drawn tail-to-tail) is θ, then $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cos(\theta)$. In particular, this means that two vectors are perpendicular if and only if their dot product is zero.

Projection: The vector projection of $\mathbf{b}$ onto $\mathbf{a}$, written $\text{proj}_{\mathbf{a}}\mathbf{b}$, is the vector formed by drawing $\mathbf{a}$ and $\mathbf{b}$ tail-to-tail, dropping a perpendicular from the tip of $\mathbf{b}$ to $\mathbf{a}$, and drawing a vector along $\mathbf{a}$ to where it meets that perpendicular.

What? Okay, maybe a picture will help:



Its length is called the component of $\mathbf{b}$ along $\mathbf{a}$, written $\text{comp}_{\mathbf{a}}\mathbf{b}$. (Why? Because if you rotated your coordinate system so that $\mathbf{a}$ sat along an axis, this would be the component of $\mathbf{b}$ in that dimension.)

The dot product allows us to compute both of these values:

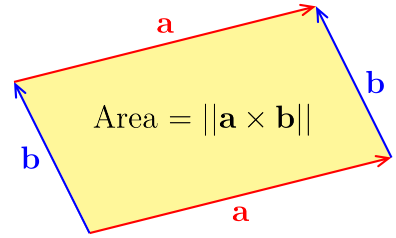
$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||} \qquad \text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||^2} \right) \mathbf{a}$$

Note that $\text{comp}_{\mathbf{a}}\mathbf{b}$ is a scalar, while $\text{proj}_{\mathbf{a}}\mathbf{b}$ is a vector.

Cross product: The cross product is *only* defined for vectors in three dimensions. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product is:

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2 \, , \, a_3 b_1 - a_1 b_3 \, , \, a_1 b_2 - a_2 b_1 \rangle$$

Note that, in contrast to the dot product, the cross product $\mathbf{a} \times \mathbf{b}$ is a vector. This vector is always perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Its direction can be found using the right-hand rule, and its magnitude is given by the formula $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \sin(\theta)$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This is equal to the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$:



If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{a} \times \mathbf{b}$ is the zero vector $\mathbf{0} = \langle 0, 0, 0 \rangle$.

Lines: Now that we know about vectors, we can accurately describe lines in three dimensions. Suppose a line passes through the point (x_0, y_0, z_0), and travels in the direction of the vector $\langle a, b, c \rangle$. Then the parametric equations for the line are:

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

We can also describe the same line with *symmetric equations*:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Planes: Suppose a plane passes through the point (x_0, y_0, z_0), and is perpendicular to $\langle a, b, c \rangle$ (called a *normal vector*). The equation for the plane can be written as:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

In the wild, you're likely to see the right side simplified, since it's just a constant:

$$ax + by + cz = d$$

Oh, and the *coordinate planes* are the xy-plane $(z = 0)$, the xz-plane $(y = 0)$, and the yz-plane $(x = 0)$.

Some things you can do with all that:

- **Find the components of a vector from one point to another.** Man, that's easy: just subtract the first point's coordinates from the second point's coordinates. The line from (a_1, a_2, a_3) to (b_1, b_2, b_3) is $\langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$.

- **Check whether two vectors are parallel.** Is one a scalar multiple of the other? Try dividing each component of one vector by the corresponding component of the other vector. Do you get the same scaling factor each time?

- **Check whether two vectors are orthogonal.** Take their dot product! Is it zero?

- **Find the angle θ between two vectors.** Take their dot product, and use the fact that $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cos(\theta)$.

- **Find out whether two lines are parallel, intersecting, or skew.** First, check whether or not they're parallel by seeing if their directional vectors are parallel.

  To see if they're intersecting, try solving the system of equations formed by setting their parametric equations equal. But wait! Make sure to use one parameter (t, probably) for one line, and another parameter (like s) for the other line. If you get a solution, then they're intersecting!

  If they're not parallel or intersecting, then they're skew lines.

- **Find a plane passing through three given points.** Pick two vectors between those points. The normal vector of the plane should be perpendicular to both of them, so find their cross product. Now you have the normal vector and a point (actually, three points) on the plane.

- **Find a plane containing a given point and a given line.** One method: find two points on that line. Now you just need a plane through those three points.
- **See where a line intersects a plane.** If the line is outside the plane and orthogonal to the plane's normal vector, then they don't intersect.

  If they do intersect, find out where by plugging the parametric equations of the line into the equation of the plane and solving for t. Plug that t back into the parametrics, and you've got the point of intersection.
- **Find the angle between two planes.** Find the angle between their normal vectors! But you probably want the acute angle, so if your answer is obtuse, find the supplement of that angle.
- **See where two planes intersect.** First, check if they're parallel. (Are their normal vectors parallel?)

  They're not? Then they intersect in a line, and that line is perpendicular to the normal vectors of both planes. Take the cross product to find its direction. Use algebra (perhaps by setting some coordinate equal to your favorite number) to find a point along that line. Now you've got a point and a direction, so you're just about done.
- **Find a line through a given point, parallel to two given planes.** Hey, this is just like the previous problem: that line should be perpendicular to both normal vectors, so you can take their cross product to find its direction.
- **Find the distance from a point (let's call it P) to a plane.** Okay, let $\mathbf{n}$ be the normal vector to the plane, and let $\mathbf{b}$ be a vector from some point (any point) on the plane to P. The distance is $\text{comp}_{\mathbf{n}}\mathbf{b}$.