Math 126 D - Spring 2015
Midterm Exam Number Two
May 19, 2015

Name: ______________________ Student ID no. : ______________

Signature: ______________________ Section: ______________

1	15	
2	12	
3	15	
4	18	
Total	60	

- This exam consists of FOUR problems on FIVE pages, including this cover sheet.
- Show all work for full credit. Show no work for zero credit.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- Write all of your work on the exam itself. If you use the back of the page, please indicate that you have done so!
- You may use a *scientific*, *non-programmable*, *non-graphing* calculator.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

1. A particle begins at the origin at time $t = 0$. At time $t = 1$, its velocity vector is $\langle 0, 4, 4\rangle$. After t seconds, its acceleration vector is $\mathbf{a}(t) = \left\langle -4, \dfrac{-16}{(t+1)^3}, \pi\sin(\pi t)\right\rangle$.

(a) **[10 points]** Write a vector function $\mathbf{r}(t)$ for the particle's position after t seconds.

If $\vec{a}(t) = \left\langle -4, \frac{-16}{(t+1)^3}, \pi\sin(\pi t)\right\rangle$, then antidifferentiate to get:

$$\vec{v}(t) = \left\langle -4t + C_1, \frac{8}{(t+1)^2} + C_2, -\cos(\pi t) + C_3 \right\rangle.$$

In order to get $\vec{v}(1) = \langle 0, 4, 4\rangle$ we have $C_1 = 4, C_2 = 2, C_3 = 3$:

$$\vec{v}(t) = \left\langle -4t + 4, \frac{8}{(t+1)^2} + 2, -\cos(\pi t) + 3 \right\rangle.$$ Antidifferentiate again:

$$\vec{r}(t) = \left\langle -2t^2 + 4t + C_4, \frac{-8}{t+1} + 2t + C_5, \frac{-1}{\pi}\sin(\pi t) + 3t + C_6 \right\rangle.$$

In order to have $\vec{r}(0) = 0$, we get $C_4 = 0, C_5 = 8, C_6 = 0$:

$$\boxed{\vec{r}(t) = \left\langle -2t^2 + 4t, \frac{-8}{t+1} + 2t + 8, \frac{-1}{\pi}\sin(\pi t) + 3t \right\rangle}$$

(b) **[5 points]** Compute the curvature of the particle's path at time $t = 3$.

$$K = \frac{|\vec{r}\,'(3) \times \vec{r}\,''(3)|}{|\vec{r}\,'(3)|^3}$$

$$\vec{r}\,'(3) = \langle -8, 2.5, 4\rangle \qquad |\vec{r}\,'(3)| = \sqrt{(-8)^2 + (2.5)^2 + (4)^2}$$

$$\vec{r}\,''(3) = \langle -4, -0.25, 0\rangle$$

$$\vec{r}\,'(3) \times \vec{r}\,''(3) = \langle 1, -16, 12\rangle \qquad |\vec{r}\,'(3) \times \vec{r}\,''(3)| = \sqrt{(1)^2 + (-16)^2 + (12)^2}$$

$$\boxed{K = \frac{\sqrt{1^2 + 16^2 + 12^2}}{\left(\sqrt{8^2 + 2.5^2 + 4^2}\right)^3}} \approx 0.025$$

2. A *right square pyramid* with base side length x and height y has surface area given by the following formula:

$$f(x,y) = x^2 + x\sqrt{x^2+4y^2}$$

(a) **[8 points]** Give the equation of the tangent plane to $z = f(x,y)$ at the point $(8,3,144)$.

$$f_x(x,y) = 2x + \sqrt{x^2+4y^2} + \frac{x^2}{\sqrt{x^2+4y^2}}$$

$\hookrightarrow f_x(8,3) = 32.4$

$$f_y(x,y) = \frac{4xy}{\sqrt{x^2+4y^2}}$$

$\hookrightarrow f_y(8,3) = 9.6$

Tangent plane through $(8,3,144)$:

$$z - 144 = f_x(8,3)(x-8) + f_y(8,3)(y-3)$$

$$\boxed{z - 144 = 32.4(x-8) + 9.6(y-3)}$$

(b) **[4 points]** A right square pyramid has a surface area of 144.402 and a height of 2.998. Use linearization to estimate the side length of the base.

$\hookrightarrow$ let's use the tangent plane from part (a)!

Plug in $z = 144.402$, $y = 2.998$:

$$144.402 - 144 = 32.4(x-8) + 9.6(2.998 - 3)$$

$\downarrow$ solve for x.

$$\boxed{x = 8.013}$$

3. **[15 points]** Let $z = f(x,y) = 3e^x(x - xy^2)$.

Find all critical points of f. Classify them as local minima, local maxima, or saddle points.

Please list the (x, y, z) coordinates for each solution.

$$f_x(x,y) = 3e^x(x-xy^2) + 3e^x(1-y^2)$$

$$f_x(x,y) = 3e^x(x-xy^2+1-y^2)$$

$$f_y(x,y) = 3e^x(-2xy)$$

$$3e^x(-2xy)=0 \Longrightarrow x=0 \quad \text{or} \quad y=0$$

(Plug into $3e^x(x-xy^2+1-y^2)=0$)

$$3(1-y^2)=0 \qquad\qquad 3e^x(x+1)=0$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$y=\pm 1 \qquad\qquad x=-1$$

Crit. pts: $(0,1,0)$ & $(0,-1,0)$ $\qquad (-1, 0, \frac{-3}{e})$

Classify!

Find $D = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2$

$$f_{xx}(x,y) = 3e^x(x-xy^2+2-2y^2)$$

$$f_{yy}(x,y) = 3e^x(-2x)$$

$$f_{xy}(x,y) = 3e^x(-2xy-2y)$$

$(0,1,0)$: $D = 0\cdot 0 - (-6)^2 < 0$ $\quad$ $(0,1,0)$ is a saddlepoint

$(0,-1,0)$: $D = 0\cdot 0 - (6)^2 < 0$ $\quad$ $(0,-1,0)$ is a saddlepoint

$(-1,0,\frac{-3}{e})$: $D = \left(\frac{3}{e}\right)\cdot\left(\frac{6}{e}\right) - 0^2 > 0$, $f_{xx} > 0$, so $\quad$ $(-1,0,\frac{-3}{e})$ is a local min

4. **[6 points each]** Compute each double integral.

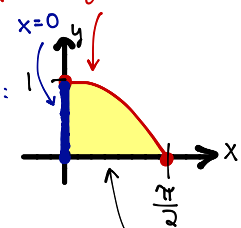
(a) $\displaystyle\iint_R x^3 e^{x^2 y}\, dA$, where $R = [5,6] \times [0,2]$.

$$\int_5^6 \int_0^2 x^3 e^{x^2 y}\, dy\, dx = \int_5^6 \left[x e^{x^2 y} \right]_0^2 dx = \int_5^6 \left(x e^{2x^2} - x \right) dx$$

$$= \left[\frac{1}{4} e^{2x^2} - \frac{1}{2} x^2 \right]_5^6 = \boxed{\left(\frac{1}{4} e^{72} - 18 \right) - \left(\frac{1}{4} e^{50} - 12.5 \right)}$$

(b) $\displaystyle\int_0^1 \int_0^{\cos^{-1}(y)} \sqrt{6 \sin(x)}\, dx\, dy$

Yuck! Let's draw and change the order of variables:



x goes from 0 to $\frac{\pi}{2}$, and for each x, y goes from 0 to $\cos(x)$.

$$\int_0^{\pi/2} \int_0^{\cos(x)} \sqrt{6 \sin(x)}\, dy\, dx = \int_0^{\frac{\pi}{2}} \left[\sqrt{6 \sin(x)}\, y \right]_0^{\cos(x)} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos(x) \sqrt{6 \sin(x)}\, dx = \int_0^1 \sqrt{6} \sqrt{u}\, du = \left[\frac{\sqrt{6} \cdot 2 u^{3/2}}{3} \right]_0^1$$

$u = \sin(x)$

$du = \cos(x) dx$

(change bounds!)

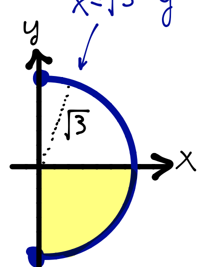
$$= \boxed{\frac{\sqrt{6} \cdot 2}{3}}$$

(c) $\displaystyle\int_{-\sqrt{3}}^0 \int_0^{\sqrt{3-y^2}} \frac{y}{1+x^2+y^2}\, dx\, dy$

Possibly useful hint: $a^2 = (a^2+1) - 1$

Note: on the exam, this $-\sqrt{3}$ was a -1, so the resulting integral is waaaay longer. Here's the intended version.

Draw it!



Convert to polar?

θ goes from $\frac{-\pi}{2}$ to 0.

r goes from 0 to $\sqrt{3}$.

$dx\, dy \to r\, dr\, d\theta$

$$\frac{y}{1+x^2+y^2} \to \frac{r \sin(\theta)}{1+r^2}$$

$$\int_{-\frac{\pi}{2}}^0 \int_0^{\sqrt{3}} \frac{r^2 + 1 - 1}{1+r^2} \sin(\theta)\, dr\, d\theta$$

split into 2 fractions then simplify

$$= \int_{-\frac{\pi}{2}}^0 \int_0^{\sqrt{3}} \left(1 - \frac{1}{1+r^2} \right) \sin(\theta)\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^0 \left[\left(r - \arctan(r) \right) \sin(\theta) \right]_0^{\sqrt{3}} d\theta = \int_{-\frac{\pi}{2}}^0 \left(\sqrt{3} - \frac{\pi}{3} \right) \sin\theta\, d\theta = \boxed{\frac{\pi}{3} - \sqrt{3}}$$